Markov switching and jump diffusion models with applications in mathematical finance

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Markov Switching and Jump Diffusion Models with Applications in Mathematical Finance

by

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Abstract

In this thesis, we study some jump diffusion models with Markov switching and transition densities for Markov switching diffusion processes with and without an absorbing barrier. We work out some analytical results, which have useful applications in mathematical finance and other related fields. The first-passage time problem for a Markov switching model is also studied and European type options and lookback options are computed in closed-form as examples to show that these models can be applied in practice. We apply optimization methods and kernel smoothing techniques to produce some important numerical results that show that jump diffusion with Markov switching models successfully capture the empirical feature of the market implied volatility of stock prices. We also use a path integral approach for a two-state Markov switching diffusion model, and it turns out that the transition probability density is a weighted average of gaussian densities for this model. As we will see in this thesis, the models can be extended to the multi-state case, but two-state models have particular applicability in the sense of economic cycles - expansion and contraction. As an interesting application, the two-state Markov switching jump diffusion model can be used for modelling insurance surplus with pricing cycles. In this case, the ruin probability is easily obtained.

Keywords: *Option Pricing, Path Integral Approach, Markov Regime Switching, Jump Diffusion, Implied Volatility, First-Passage Time, Risk Process, Ruin Probability*

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Chapter 1

Introduction

1.1 Motivation

In financial modelling, the Black-Scholes (B-S) model plays an important role in pricing risky assets, and has laid the foundation for many option pricing frameworks. This famous model has proven useful as a theoretical basis in mathematical finance since there are many analytical solutions for the price of most types of options under this model, such as barriers, lookbacks and other exotic options. However, the Black-Scholes model fails to capture important empirically observed features of stock price returns and market implied volatility surfaces. This has brought about much interest in the development of alternative models. As is well known in the B-S model the volatility of the asset price is constant so that it cannot fully represent the dynamic or stochastic behavior of the corresponding asset price movements. Instead, recent research focuses on the nonlinear stochastic volatility models. Campolieti [4,5] and his research group are studying new families of integrable continuous diffusion models. This work focuses on transition densities for certain types of non-linear stochastic differential equations (SDEs). Using these transition densities, one is able to work on option pricing, first-passage time problems, etc. One main reason for studying such alternative models is to try perfectly recover the skewness or smile features of the market implied volatility. Not many models have closed-form solutions, particularly for exotic options, which forces us to do intensive numerical calculations. Jump diffusion models are often studied and some [2,21] have very good results for model calibration. For this type of model, one assumes that the underling risky assets satisfy a linear stochastic differential equation (the continuous component) plus a compound Poisson process (the jump or discontinuous component). Kou [2] studies a jump diffusion model for option pricing. Basically, Kou's model assumes that the percentage of jump size follows a double exponential with certain mixture probability. The result he has investigated shows that jump diffusion models can be very useful in option pricing. Due to the analytical solution for the pricing density for this particular jump diffusion process, Kou also proposes the first-passage time problem for this jump diffusion model [3]. There are many advantages for the jump diffusion model. First of all, the jump diffusion model can capture important empirical features, such as volatility smile or skewness. Secondly, jump processes can explain the incompleteness of the financial market, which means the real market cannot be perfectly hedged in a short time. Finally, financial market disasters tend to be important issues in financial analysis, and modelling financial disasters using a jump process is reasonable. Models that incorporate significant jumps in market prices or market returns can explain the market uncertainty and market disasters.

A Markov chain has the basic property that the process is only dependent on the previously observed value and is independent with respect to all other past history. There are some important advantages in using Markov models. One of these benefits is that we can easily work out the joint density function for many processes due to the independency between the current process and its history. Therefore, the transition matrix for the process after a certain time point is attainable. Applying these conditional transition probabilities, one can obtain the probability density of a process with hidden Markov chain explicitly. Duan [1] has studied a Markov switching model for option pricing in a discrete time setting and he has shown that a Markov switching model could describe the behavior of volatility regimes. Volatility regimes are important factors inherent in an economy. What we observe in this phenomenon is the different variation of the rate of return for financial assets during some specific time period. Typically, there are several regimes of variation. (We can see this from Figure 1.4, which is the time series of the log return of weekly TSE 100 COMPX.) In Figure 1.1, we simulate and compare four different regimes of volatility for geometric Brownian motion. They show the different variation of asset prices, i.e., the higher the volatility, the higher the variation of asset prices (Note: the different vertical scales are chosen to more visibly reflect the relative price differences). In the market, the mean and variance of the market price or rate of return are normally time dependent. Therefore, in the work of financial modelling, it is not a good idea to always assume a constant rate of diffusion. We should at least consider a reasonable number of regimes for diffusion to successfully capture the true market behavior.

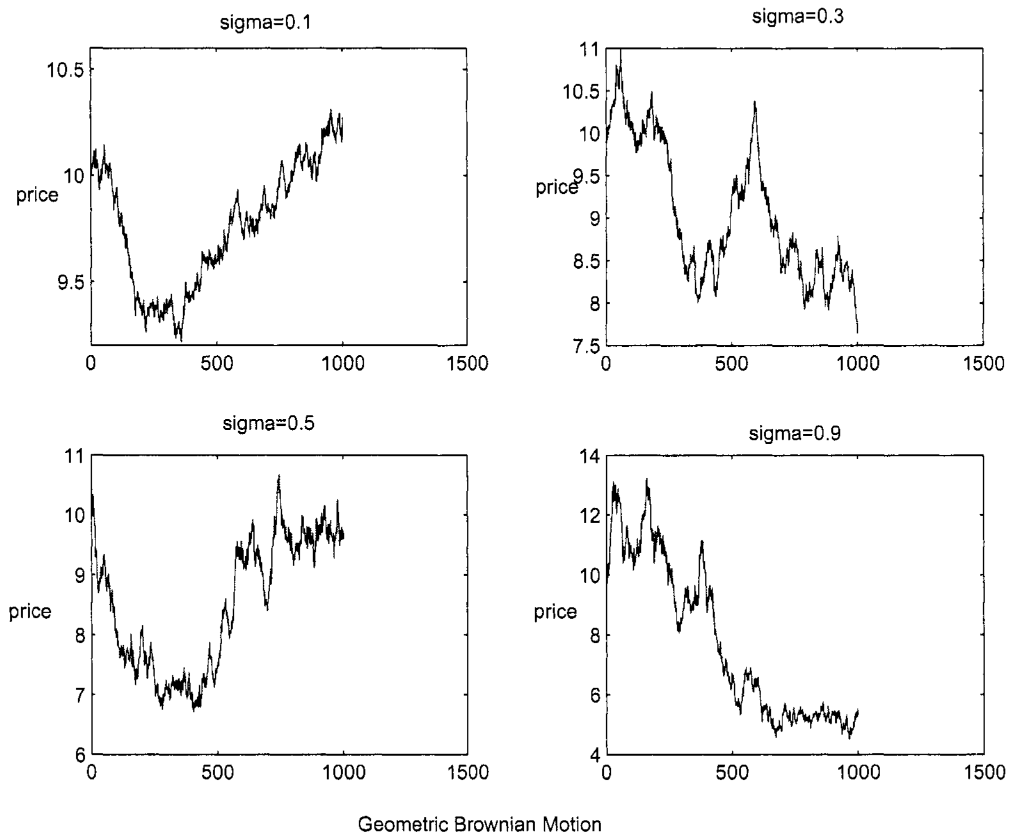

Figure 1.1: Simulated Paths of Geometric Brownian Motions with Different Diffusion Term.

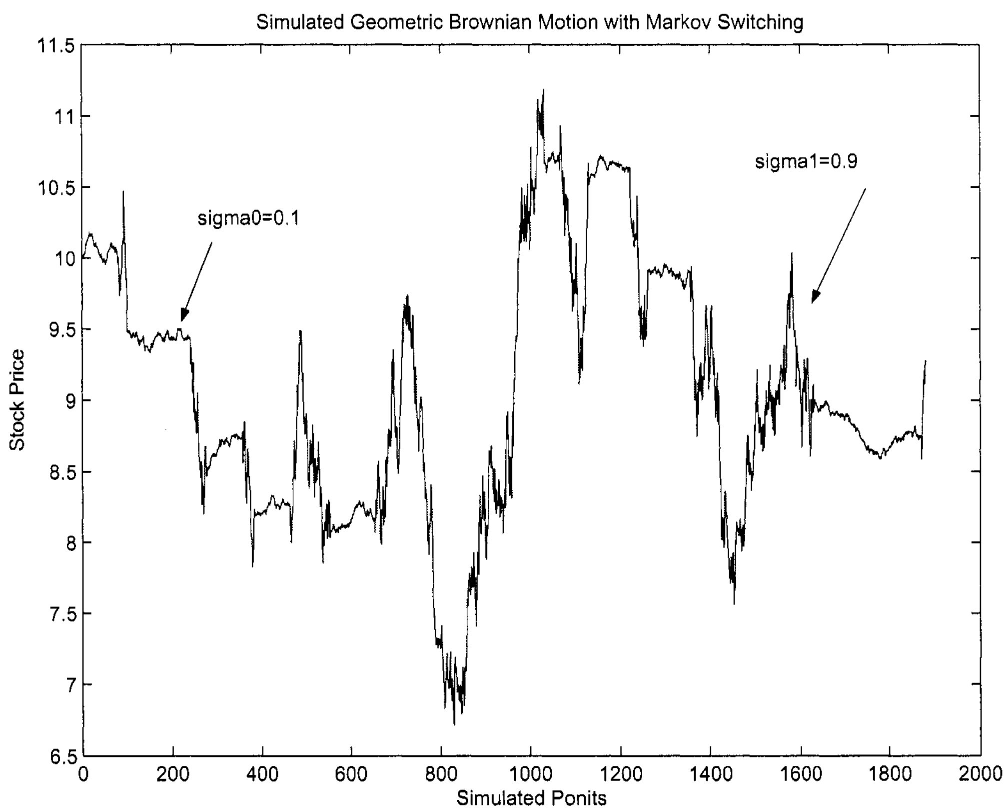

Figure 1.2: Simulated Paths of Markov Switching Geometric Brownian motion.

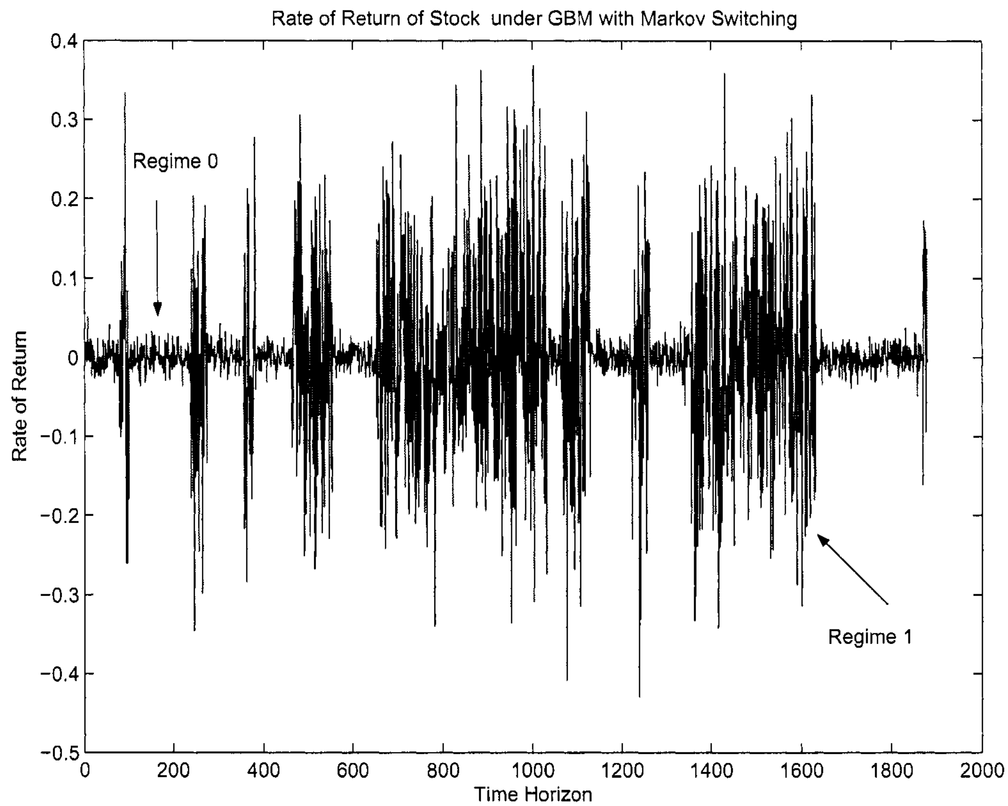

Figure 1.3: Rates of Return under Markov Switching Geometric Brownian motion.

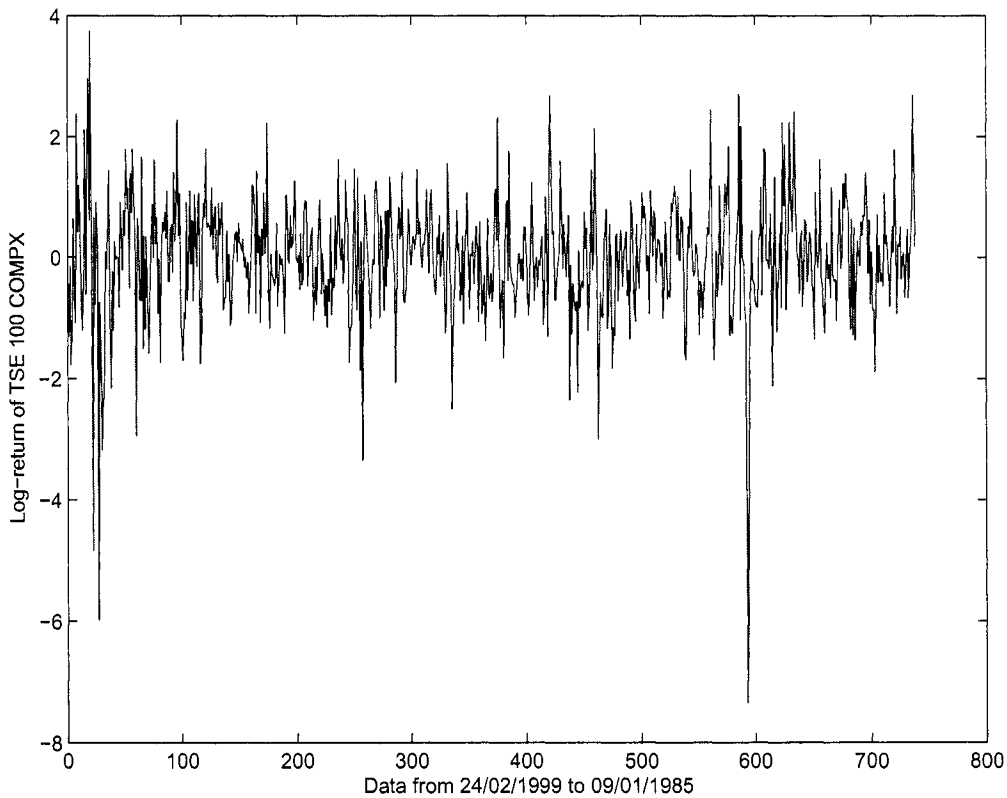

Figure 1.4: Time Series of Weekly Log-return for TSE 100 COMPX.

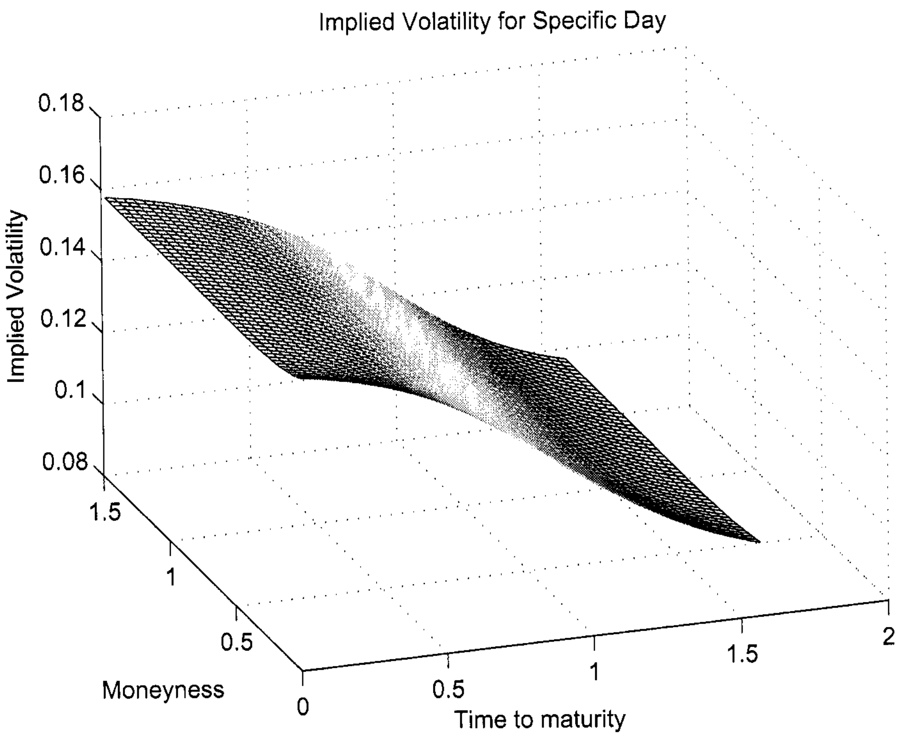

Figure 1.5: Surface of Implied Volatility of Daily S&P 500 Options.

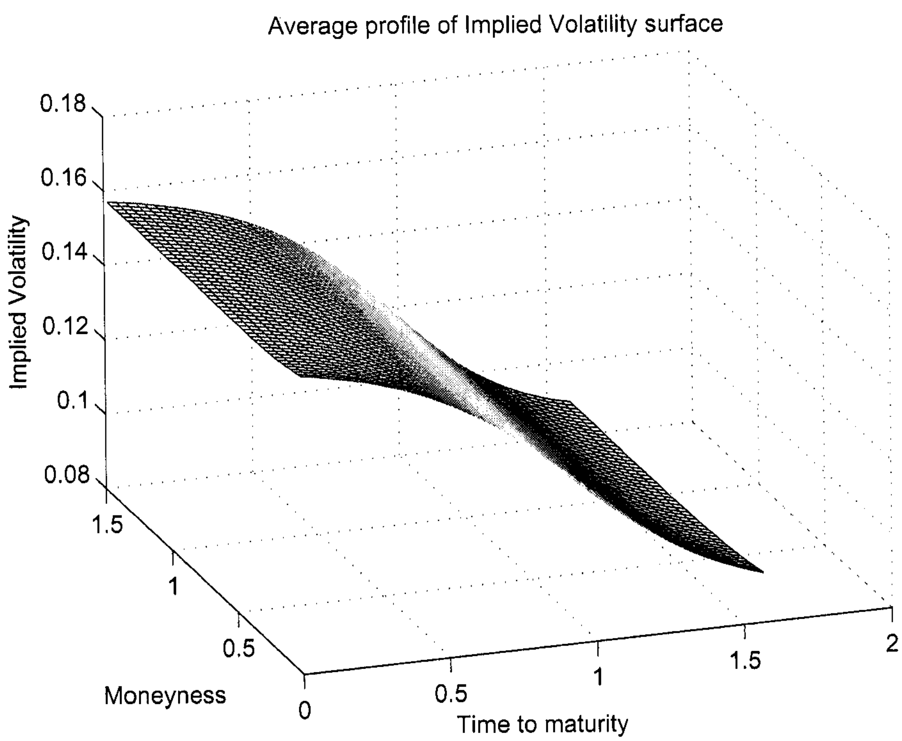

Figure 1.6: Average Surface of Implied Volatility of S&P 500 Options from 04-Jan-1993 to 15-Jan-1993.

In an economy there usually exists two business cycles - expansion and contraction. In the cycle of expansion the investor is expecting a higher rate of return on a financial asset. On the other hand, lower rates of return will occur during the period of a business contraction. In Figure 1.2, we simulate a path from a geometric Brownian motion with Markov switching. There are two regimes in the process, one has the lower rate of volatility (σ=0.1) and another has the higher rate of volatility (σ=0.9). Obviously, the higher payoff of the asset investment will accompany a higher risk, this can be seen from the Figure 1.3. Market risk is normally measured by the standard deviation of the market price. It is important that we consider the volatility for different periods of the business cycle and the switching of these two business cycles is assumed in the Markov environment. Therefore, combining a jump diffusion process and Markov switching into the model for asset price movements is our main theme in this thesis. There are many quantities that can be obtained analytically if we can find the transition density for the Markov switching jump diffusion model explicitly. In particular, we can price many types for options, such as European call and put options, lookback options, etc.

This thesis is organized as follows: In Chapter One, we give some useful background. In Chapter Two, we investigate the new pricing models and their associated transition densities. Option pricing under those models is also studied in this chapter. In Chapter Three, we obtain some important numerical applications of the models we propose. In Chapter Four, we introduce an interesting application to actuarial science using a similar idea as the Markov switching jump diffusion model and study the ruin probability problem using the new model, which is a core problem in actuarial science.

1.2 Some Useful Background

1.2.1 Brownian Motion

Consider the process X_t that satisfies the following SDE

$$dX_t = rdt + \sigma dB_t, X_0 = 0. \tag{1.1}$$

X_t is the so-called Brownian motion with drift r. $B_{t\geq 0}$ is a standard Brownian motion w.r.t. some filtration $(F_t)_{t\geq 0}$ in probability space (P, F_t, Ω). The transition probability density function for X_t is a gaussian density given by

$$g_{0,r}(x, x_0, t) = \frac{e^{-(x-x_0-rt)^2/2\sigma^2 t}}{\sigma\sqrt{2\pi t}}. \tag{1.2}$$

This Gaussian density is time dependent and the variance of X_t is a linear function with respect to time t. Brownian motion gives us a constant rate of variation, which may not be realistic from a practical point of view. This is one main reason we are going to investigate an extension of the model that is more reasonable.

The Reflection Principle for Brownian Motion with Drift

Let X_t denote the Brownian motion starting at $x_0 < x_H$ at initial time t_0 with an *upper* absorbing barrier x_H. Let $\tilde{X}_t$ denote the free path Brownian motion. We would like to set out to compute the probability that a path X_s, $0 \leq s \leq t$, starting at $x_0 < x_H$ has the value of x or less at time t, given that $x < x_H$. This probability is given by

$$P(X_t \leq x) = P(\tilde{X}_t \leq x, \sup_{0\leq s\leq t} \tilde{X}_s < x_H). \tag{1.3}$$

From the first principle of total probability of events

$$\{\tilde{X}_t \leq x\} = \{\tilde{X}_t \leq x, \sup_{0\leq s\leq t} \tilde{X}_s < x_H\} \cup \{\tilde{X}_t \leq x, \sup_{0\leq s\leq t} \tilde{X}_s \geq x_H\}. \tag{1.4}$$

Hence

$$\begin{aligned} P(\tilde{X}_t \le x) &= P(\tilde{X}_t \le x, \sup_{0\le s\le t} \tilde{X}_s < x_H) + P(\tilde{X}_t \le x, \sup_{0\le s\le t} \tilde{X}_s \ge x_H) \\ &= P(\tilde{X}_t \le x) + P(\tilde{X}_t \le x, \sup_{0\le s\le t} \tilde{X}_s \ge x_H). \end{aligned} \tag{1.5}$$

Since a free Brownian path at x_H at time t_H subsequently attaining the value x at terminal time t is the same as that for a reflected path starting at x_H at time t_H and attaining the value $2x_H - x$ at terminal time t, then for both paths this probability density is

$$g_{0,r}(x, x_H, s) = g_{0,r}(2x_H - x, x_H, s) \tag{1.6}$$

where s is the remaining time from the time t_H to terminal time t. Using this result, the second term in (1.5) becomes

$$P(\tilde{X}_t \le x, \sup_{0\le s\le t} \tilde{X}_s \ge x_H) = P(\tilde{X}_t \ge 2x_H - x). \tag{1.7}$$

Substituting this result into equation (1.5) gives

$$P(X_t \le x) = P(\tilde{X}_t \le x) - P(\tilde{X}_t \ge 2x_H - x). \tag{1.8}$$

1.2.2 Geometric Brownian Motion

The price of a stock or other risky asset is usually modeled by the so-called geometric Brownian motion (GBM) given as follows

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t \tag{1.9}$$

where S_t is the positive price of stock or other risky asset, r is the riskless interest rate of return, σ is the volatility of the return and $B_{t\ge 0}$ is standard Brownian motion under the risk-neutral measure. In the case of the physical measure, r should be replaced by physical rate of return μ. The strong solution of S_t for geometric Brownian motion is

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t} \tag{1.10}$$

where S_0 is the stock price at time zero. We also can see that the strong solution of the SDE is normally not analytically attainable if we have non-linear volatility except for some special cases. We should realize from Equation (1.10) that for fixed t, the asset price is a random variable having a lognormal distribution and time dependent mean and variance. Due to some important properties of geometric Brownian motion, we consider an extension of this model. In our modelling, we are exploring some Markov switching geometric Brownian motion with certain jump processes.

GBM without Absorption

For geometric Brownian motion with free path $S_t \in (0, \infty)$, a general relationship between the S-space and the x-space (standard Brownian motion) densities is obtained using the map $X(S) := \log(S)$ giving:

$$\begin{aligned} U_0(S, S_0, t) &= \frac{1}{S} u_{\mu - \frac{1}{2}\sigma^2}(X(S), X(S_0), t) \\ &= \frac{1}{\sigma S \sqrt{2\pi t}} e^{-[\log(S/S_0) - (\mu - \sigma^2/2)t]^2 / 2\sigma^2 t} \end{aligned} \tag{1.11}$$

for all S, $S_0 \in (0, \infty)$ and $t > 0$. Here the notation u_μ refers to a density for standard Brownian motion with drift μ, which is given in the form of (1.2) for the risk-neutral measure $\mu = r$.

GBM with Absorption

For the case of a single barrier at $S = H$, using Equation (1.8), the kernel in (1.11) becomes

$$\begin{aligned} U(S, S_0, t) &= \frac{1}{S}\left[g_{0,\mu - \frac{1}{2}\sigma^2}(\log S, \log S_0, t) - (H/S_0)^{\frac{2\mu}{\sigma^2} - 1} g_{0,\mu - \frac{1}{2}\sigma^2}(\log S, \log(H^2/S_0), t) \right] \\ &= U_0(S, S_0, t) - (H/S_0)^{\frac{2\mu}{\sigma^2} - 1} U_0(S, H^2/S_0, t). \end{aligned} \tag{1.12}$$

This formula is valid for either an upper or lower barrier where $S, S_0 \in (0, H]$ or $S, S_0 \in [H, \infty)$, respectively.

1.2.3 Compound Poisson Process

Definition 1. *A random process*

$$Y_t = \sum_{i=0}^{N(t)} X_t \tag{1.13}$$

is called compound Poisson process if X_t are i.i.d distributed and $N(t)$ is a Poisson random variable.

More generally, if $N(t)$ is a point process, then Y_t is called compound process. A compound Poisson process is often used as a jump process, however, for most choices of distributions of X_t, the compound distributional values can only be obtained numerically. For certain choices of distributions of X_t, simple analytic results are available, thus reducing the computational problems considerably. For instance, if X_t follows an exponential distribution with mean θ, the corresponding c.d.f. of Y_t is

$$\begin{aligned} F_Y(x) &= 1 - e^{-x/\theta} \sum_{j=0}^{\infty} \frac{(x/\theta)^j}{j!} \sum_{n=j+1}^{\infty} p_n \\ &= 1 - e^{-x/\theta} \sum_{j=0}^{\infty} P_j \frac{(x/\theta)^j}{j!} \end{aligned} \tag{1.14}$$

where $P_j = \sum_{n=j+1}^{\infty} p_n$ for $j = 0, 1, \cdots$, $p_n = P(N = n)$ and $x \geq 0$. For details on the compound model, one can refer to a loss model textbook [7]. Since we just consider an extension of GBM and our goal is to end up with some analytical results, it is very important to restrict our distribution of jump size to be of a certain type.

1.2.4 Discrete-Time Markov Chains

For a process $\{X_n, n \geq 0\}$, with state space $S = \{0, 1, \cdots\}$ and $\{a_k\}_{k \geq 0}$, where $\{a_k\}_{k \geq 0}$ are such that $0 \leq a_k \leq 1, \forall\, k \geq 0$ and $\sum_{i=0}^{\infty} a_i = 1$ (This is initial distribution), we let $P = (p_{ij})_{i,j \geq 0}$ be a stochastic matrix. That is

$$P = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0i} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1i} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \tag{1.15}$$

such that $0 \leq p_{ij}, \forall i, j \geq 0$ and $\sum_{j=0}^{\infty} p_{ij} = 1, \forall i \geq 0$.

Definition 1. *Process $\{X_n, n \geq 0\}$ is called a Markov Chain if $P(X_0 = i_0, X_1 = i_1, \cdots, X_k = i_k) = a_{i0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k}$ for all $k \geq 0$ and $(i_0, \cdots, i_k) \in S$.*

Proposition 1. *a Markov Chain $\{X_n, n \geq 0\}$ satisfies the following properties:*

- $P(X_0 = i) = a_i, \forall i \geq 0,$
- $P(X_{n+1} = j | X_n = i) = p_{ij}, \forall n \geq 0,$
- $P(X_{n+1} = j | X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i) = p_{ij},$

for all $i_0, \cdots, i_{n-1}, i, j \in S$.

1.2.5 Continuous-Time Finite-State Markov Chains

We consider a random process $\{X(t), t \geq 0\}$ with index set $[0, \infty)$ and a state space which is a subset of the integers. Given that the process is in state i, we shall assume that the holding time that process will be staying in state i follows an exponential distribution with mean $\lambda_i > 0$. Given the information of the current state, the holding times are independent random variables. Therefore, we have the following important result for this random process:

$$P(X_{t+\tau} = n, 0 < \tau < s \mid X_t = n) = e^{-\lambda_n s}, n \in \{0, 1, \cdots\}. \tag{1.16}$$

This is called the memoryless property. For the two-state continuous Markov chain, we have the following solution for $P(t)$, which is the transition matrix after time t:

$$P(t) = \begin{pmatrix} 1 - \int_0^t \lambda_0 e^{-(\lambda_0+\lambda_1)s} ds & \int_0^t \lambda_0 e^{-(\lambda_0+\lambda_1)s} ds \\ \int_0^t \lambda_1 e^{-(\lambda_0+\lambda_1)s} ds & 1 - \int_0^t \lambda_1 e^{-(\lambda_0+\lambda_1)s} ds \end{pmatrix}$$

$$= \frac{1}{\lambda_0 + \lambda_1} \begin{pmatrix} \lambda_1 + \lambda_0 e^{-(\lambda_0+\lambda_1)t} & \lambda_0 - \lambda_0 e^{-(\lambda_0+\lambda_1)t} \\ \lambda_1 - \lambda_1 e^{-(\lambda_0+\lambda_1)t} & \lambda_0 + \lambda_1 e^{-(\lambda_0+\lambda_1)t} \end{pmatrix}. \tag{1.17}$$

The way to understand the transition matrix $P(t)$ is to treat the two states as an on-off system. Whenever a state is on, it stays that way for a random length of time with exponential density with mean $1/\lambda_0$. When switching occurs, the new state off, lasting for a random length of time with exponential density with mean $1/\lambda_1$, is started immediately. For the transition matrix $P(t)$, as $t \longrightarrow \infty$, the limiting transition matrix becomes

$$P(t) = \frac{1}{\lambda_0 + \lambda_1} \begin{pmatrix} \lambda_1 & \lambda_0 \\ \lambda_1 & \lambda_0 \end{pmatrix}. \tag{1.18}$$

Chapter 2

Pricing Models

In this chapter, we investigate an extension of geometric Brownian motion. In the first part of this chapter, we introduce the jump diffusion models with Markov switching. In order to ensure that the models have closed-form solutions for the associated transition density, we restrict our choices of percentage of jump size to certain distribution. The exponential, constant and lognormal percentage of jump size distributions are studied in detail. The latter part of the chapter focuses on transition densities for the Markov switching diffusion process and the first-passage time problem. Since we have the closed-form solution for the transition density of the Markov switching diffusion model, as well as the distribution of the minimum and maximum of the process, we then also study the lookback option and give its pricing formula as well.

2.1 A Jump Diffusion Model

Jump diffusion models are considered for pricing risky assets since these models display good calibration results with respect to real data. These models normally have constant drift rate and diffusion term. However, it is not necessary to have this restriction, we still can consider the case of a non-linear SDE model with jumps. But the strong solution of S_t for the corresponding SDE may not be readily attainable. Instead, we should generally consider some numerical approximation method for solving the SDE and a simulation method for the pricing problem. Many recent researchers consider the following model for the movement of the underlying risky asset:

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t + d(\sum_{i=1}^{N(t)} Y_J) \tag{2.1}$$

where Y_J is a percentage of jump size and has a certain identical independent distribution, $N(t)$ is a counting process. In particular, it is a Poisson random variable, i.e., $P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, but we can also consider $N(t)$ as a negative binomial distribution. Here, Y_J is usually a random variable. We should realize that for the case of $Y_J{=}\rho$, where ρ is a constant, the model reduces to the famous Merton jump diffusion model. In this thesis, we assume that Y_J is an i.i.d random variable with a specific distribution such as lognormal. Current research suggests that a jump diffusion model can capture the empirical phenomena that reflects the skewness or smiles of the implied volatility. Readers can refer to Kou's work [1]. It is important to know what kind of jump size distribution is suitable in order to have an explicit solution for some pricing problems. The choices of jump size distribution are limited. Only the exponential types of distribution are potential candidates since we have to work out the convolution of many i.i.d random variables. The essential key point is that we cannot have a heavy tail distribution for the jump process, otherwise the associated convolution does not exist.

2.2 Black-Scholes Model with Markov Switching

Since there exist two business cycles in an economy, expansion and contraction, some researchers have considered an extension of Black-Scholes to reflect this economic phenomenon. The basic model is as follows:

$$\frac{dS_t}{S_t} = rdt + \sigma_{\epsilon_t} dB_t \tag{2.2}$$

where ϵ_t={0,1}. It is a two-state Markov process with certain switching probability. These states represent the associated volatility regimes. In particular, we assume that the probability of staying in a state given that the current state is n follows an exponential distribution that is characterized by the conditional probability

$$P(\epsilon_{t+\tau} = n, 0 < \tau < s \mid \epsilon_t = n) = e^{-\lambda_n s}, \;\; n \in \{0,1\} \tag{2.3}$$

where λ_n corresponds to the intensity of holding in state n. This implies that the switching process follows a Poisson distribution with intensity λ_n for state n.

2.3 Two-State Markov Switching Jump Model

Consider the following model for stock (or any other type of asset) movement within the risk neutral measure:

$$\frac{dS_t}{S_{t-}} = (r - \lambda\mu_J)dt + \sigma_{\epsilon_t} dB_t + d(\sum_{i=1}^{N(t)}(V_i - 1)) \tag{2.4}$$

where V_i is the percentage jump random variable. In this new model, we consider the jump process and different regimes for stock movement. S_{t-} denotes the stock price just before a jump at time t. The jump is going to reflect shocks in the stock and the business cycles. Here we only consider the different regimes of the stock volatility to ensure the martingale property of the model. Ignoring the different regimes of the drift is hence necessary. This implies that the riskless rate of the market account is unique. If we were to consider a case that is similar to Black-Scholes model with Markov switching by changing r to be a state dependent variable, then we would have difficulty to solve the option pricing problem in closed-form. We should realize that the drift coefficient $(r - \mu\lambda_J)$ of the model is shifted by the expectation of the compound Poisson process. Hence the entire process is such that the discounted asset price has the martingale property.

2.4 Option Pricing for Markov Switching with Jumps

2.4.1 Lognormal Percentage of Jump Size

In this section, we will make an assumption that $Y = \ln V_i$ is an i.i.d. normal random variable, $(V_i - 1)$ is the percentage jump size that is assumed to be i.i.d. lognormal with mean μ_J and variance σ_J^2. i.e., $Y \sim N\left(\ln(1+\mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right)$. It is easy to verify from Itô's Lemma that

$$S_t = S_0 \exp\{\int_0^t (r - \lambda\mu_J - \frac{1}{2}\sigma_{\epsilon_s}^2)ds + \int_0^t \sigma_{\epsilon_s} dB_s + \sum_{i=1}^{N(t)} \ln V_i\} \tag{2.5}$$

solves (2.4). Under the risk-neutral measure the discounted asset price satisfies the martingale property:

$$
\begin{aligned}
E\Big[e^{-rT}S_T|S_{t=0}=S_0\Big] &= E\Big[S_0\exp\{\int_0^T(r-\lambda\mu_J-\frac{1}{2}\sigma_{\epsilon_s}^2)ds+\int_0^T\sigma_{\epsilon_s}dB_s+\sum_{i=1}^{N(T)}\ln V_i\}\Big] \\
&= S_0E\Big[\exp\{\sum_{i=1}^{N(T)}\ln V_i-\lambda\mu_J T\}\Big]E\Big[\exp\{-\frac{1}{2}\int_0^T\sigma_{\epsilon_s}^2ds+\int_0^T\sigma_{\epsilon_s}dB_s\}\Big] \\
&= S_0E\Big[\exp\{\sum_{i=1}^{N(T)}\ln V_i-\lambda\mu_J T\}\Big] \\
&= S_0e^{-\lambda\mu_J T}E\Big[\exp\{\sum_{i=1}^{N(T)}\ln V_i\}\Big] \\
&= S_0e^{-\lambda\mu_J T}E_N\Big[E[\exp\{\sum_{i=1}^{n}\ln V_i\}\mid N(T)=n]\Big] \\
&= S_0e^{-\lambda\mu_J T}E_N\Big[E[\prod_{i=1}^{n}V_i\mid N(T)=n]\Big] \\
&= S_0e^{-\lambda\mu_J T}E_N\Big[\prod_{i=1}^{n}E[V_i\mid N(T)=n]\Big] \\
&= S_0e^{-\lambda\mu_J T}E_N\Big[(\mu_J+1)^N\Big] \\
&= S_0e^{-\lambda\mu_J T}e^{-\lambda T}(\sum_{n=0}^{\infty}\frac{(\lambda T(\mu_J+1))^n}{n!}) \\
&= S_0e^{-\lambda\mu_J T}e^{-\lambda T}e^{\lambda T\mu_J+\lambda T}=S_0.
\end{aligned}
$$

Here $E_N[\cdot]$ represents the expectation w.r.t. the Poisson random variable. So the discounted stock price is indeed a martingale. *Conditioning* on $N(t)=n$, $\sum_{i=1}^n \ln V_i \sim N(n\ln(\mu_J+1)-\frac{n\sigma_J^2}{2}, n\sigma_J^2)$. We have the following solution for S_t denoted by the new random variable (for given $n\geq 1$):

$$X_t=S_0\exp\{\int_0^t(r-\lambda\mu_J-\frac{1}{2}\sigma_{\epsilon_s}^2)ds+\int_0^t\sigma_{\epsilon_s}dB_s+\sum_{i=1}^{n}\ln V_i\} \tag{2.6}$$

taking $\ln(X_t)=Y_t$, we have $\ln(X_t)=Y_t=\ln S_0+\int_0^t(r-\lambda\mu_J-\frac{1}{2}\sigma_{\epsilon_s}^2)ds+\int_0^t\sigma_{\epsilon_s}dB_s+\sum_{i=1}^n\ln V_i$. Y_t is therefore a normal random variable. Consider a European call option conditional on $N(T)=n$, the current economic state being i, and spot price S_0, then the corresponding option price is

$$
\begin{aligned}
& C_n(T, K, r, \sigma_0, \sigma_1, \mu_J, \sigma_J, \lambda, i) \\
= & E\Big[e^{-rT}(X_T - K)_+ \mid \epsilon_0 = i\Big] \\
= & E_i\Big[E\left[(e^{-rT}(X_T - K)_+ \mid T_i\right] \mid \epsilon_0 = i)\Big] \\
= & e^{-rT}\int_0^\infty \int_0^T \frac{y}{y+K}\phi(ln(y+K), m(t), v(t)) f_i(t,T) dt dy. \qquad (2.7)
\end{aligned}
$$

Here $f_i(t, T)$ is the density function of T_i where T_i is the total time between 0 and the time-to-maturity T during which $\epsilon_t = 0$. $\phi(x, m(t), v(t))$ is the normal density, with mean

$$m(t) = \ln(S_0) + n\ln(\mu_J + 1) - n\frac{\sigma_J^2}{2} - \lambda\mu_J T + \frac{1}{2}(\sigma_1^2 - \sigma_0^2)t + (r - \frac{1}{2}\sigma_1^2)T \quad (2.8)$$

and variance

$$v(t) = n\sigma_J^2 + (\sigma_0^2 - \sigma_1^2)t + \sigma_1^2 T. \qquad (2.9)$$

Furthermore, the European call option value given current economic state i in this model is obtained by

$$
\begin{aligned}
C(\cdot, i) &= E_n\Big[C_N(\cdot, i)\Big] \\
&= \sum_{n=0}^{\infty} C_n(\cdot, i) p(N_t = n)
\end{aligned}
$$

where $p(N_t = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$. Actually, $C_n(\cdot, i)$ of (2.7) can be simplified to the following form

$$C_n(\cdot, i) = \int_0^T \left[e^{-rT + v(t)/2 + m(t)}\Phi(d_1(t)) - Ke^{-rT}\Phi(d_2(t))\right] f_i(t,T) dt \qquad (2.10)$$

where $d_1(t) = \frac{m(t)+v(t)-\ln K}{\sqrt{v(t)}}$, $d_2 = \frac{m(t)-\ln K}{\sqrt{v(t)}}$ and $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2} dy$.

Theorem 1. *Assume that the underlying asset is modeled by*

$$\frac{dS_t}{S_{t-}} = (r - \lambda\mu_J)dt + \sigma_\epsilon dB_t + d(\sum_{i=1}^{N(t)}(V_i - 1)) \qquad (2.11)$$

where $Y = \ln V_i$ *has a normal distribution with mean* $\ln(\mu_J + 1) - \frac{\sigma_J^2}{2}$ *and variance* σ_J^2. ϵ_t, *is a two-state Markov process with switching intensities* λ_0 *and* λ_1. *Then, the European call option has value*

$$C = \sum_{i=0}^{1} \sum_{n=0}^{\infty} \pi(i) C_n(\cdot, i) \frac{e^{-\lambda T}(\lambda T)^n}{n!} \tag{2.12}$$

where

$$C_n(\cdot, i) = \int_0^T \left[e^{-rT+v(t)/2+m(t)} \Phi(d_1(t)) - K e^{-rT} \Phi(d_2(t)) \right] f_i(t, T) dt \tag{2.13}$$

$$d_1(t) = \frac{m(t) + v(t) - lnK}{\sqrt{v(t)}}, \tag{2.14}$$

$$d_2(t) = d_1(t) - \sqrt{v(t)}, \tag{2.15}$$

$\pi(i)$ *is the initial probability of state* i, *and* T *is time-to-maturity.*

Proof of the Theorem 1. From equation (2.7)

$$C_n(\cdot, i) = e^{-rT} \int_0^{\infty} \int_0^T \frac{y}{y+K} \phi(ln(y+K), m(t), v(t)) f_i(t, T) dt dy. \tag{2.16}$$

Let $ln(y + K) = x$, then $e^x = y + K$ and $\frac{1}{y+K} dy = dx$. Then C_n can be rewritten as

$$e^{-rT} \int_0^T f_i(t, T) \left[\int_0^{\infty} \frac{y}{y+K} \phi(ln(y+K), m(t), v(t)) dy \right] dt. \tag{2.17}$$

The inner integral gives:

$$
\begin{aligned}
& \int_0^\infty \frac{y}{y+K}\phi(ln(y+K), m(t), v(t))dy \\
= & \int_0^\infty \frac{1}{\sqrt{2\pi v(t)}}\frac{y}{y+K}e^{-\frac{(ln(y+K)-m(t))^2}{2v(t)}}dy \\
= & \int_{lnK}^\infty \frac{1}{\sqrt{2\pi v(t)}}(e^x - K)e^{\frac{-(x-m(t))^2}{2v(t)}}dx \\
= & \int_{lnK}^\infty \frac{1}{\sqrt{2\pi v(t)}}e^{x-\frac{(x-m(t))^2}{2v(t)}}dx - K\int_{\ln K}^\infty \frac{1}{\sqrt{2\pi v(t)}}e^{-\frac{(x-m(t))^2}{2v(t)}}dx \\
= & \int_{lnK}^\infty \frac{1}{\sqrt{2\pi v(t)}}e^{\frac{-x^2+2m(t)x-m(t)^2+2v(t)x}{2v(t)}}dx - K(1-\Phi(\ln K, m(t), v(t))) \\
= & \int_{lnK}^\infty \frac{1}{\sqrt{2\pi v(t)}}e^{\frac{-(x-(m(t)+v(t)))^2+(m(t)+v(t))^2-m(t)^2}{2v(t)}}dx - K(1-\Phi(\ln K, m(t), v(t))) \\
= & e^{\frac{v(t)}{2}+m(t)}\Big[1-\Phi(\ln K, m(t)+v(t), v(t))\Big] - K\Big[1-\Phi(\frac{\ln K - m(t)}{\sqrt{v(t)}})\Big] \\
= & e^{\frac{v(t)}{2}+m(t)}\Big[\Phi(\frac{m(t)+v(t)-\ln K}{\sqrt{v(t)}})\Big] - K\Big[\Phi(\frac{m(t)-\ln K}{\sqrt{v(t)}})\Big]. \qquad (2.18)
\end{aligned}
$$

Hence,

$$
C_n(\cdot, i) = \int_0^T \left(e^{-rT+\frac{v(t)}{2}+m(t)}\Phi(\frac{m(t)+v(t)-\ln K}{\sqrt{v(t)}}) - e^{-rT}K\Phi(\frac{m(t)-\ln K}{\sqrt{v(t)}})\right) f_i(t,T)dt.
$$

Therefore

$$
C(\cdot, i) = E_n\Big[C_N(\cdot, i)\Big] = \sum_{n=0}^\infty C_n(\cdot, i)\frac{(\lambda T)^n e^{-\lambda T}}{n!}
$$

Taking the expectation w.r.t initial state i gives:

$$
C = \sum_{i=0}^1 \sum_{n=0}^\infty \pi(i) C_n(\cdot, i)\frac{e^{-\lambda T}(\lambda T)^n}{n!}. \qquad (2.19)
$$

□

Up to now, we haven't specified our density function of $f_i(t,T)$. In order to fully describe our model, the specification of this density is necessary. This density function is studied by Fuh [11, 15], for details on the derivation of this density function, the reader can refer to Fuh's work. The form of density is given by

$$
\begin{aligned}
f_0(t,T) &= e^{-\lambda_0 T}\delta_0(T-t) + e^{-\lambda_1(T-t)-\lambda_0 t}[\lambda_0 I_0(2(\lambda_0\lambda_1 t(T-t))^{1/2}) + \\
&\quad (\frac{\lambda_0\lambda_1 t}{T-t})^{1/2} I_1(2(\lambda_0\lambda_1 t(T-t))^{1/2})]
\end{aligned}
$$

$$
\begin{aligned}
f_1(t,T) &= e^{-\lambda_1 T}\delta_0(T-t) + e^{-\lambda_1(T-t)-\lambda_0 t}[\lambda_1 I_0(2(\lambda_1\lambda_1 t(T-t))^{1/2}) + \\
&\quad (\frac{\lambda_0\lambda_1 (T-t)}{t})^{1/2} I_1(2(\lambda_0\lambda_1 t(T-t))^{1/2})],
\end{aligned}
$$

where $\delta(t) = 1$ if and only if $t = 0$, otherwise $\delta(t) = 0$, and I_0 and I_1 are the modified Bessel functions of the first kind of order 0 and 1, respectively.

2.4.2 Exponential Percentage of Jump Size

Let us revisit the model in (2.4) and assume the $\ln(V_i) = \xi_i, i \geq 0$ follow an exponential distribution with mean $\frac{1}{\theta}$. Then, the compound Poisson process with exponential jump size follows a gamma distribution with parameters n and θ. Conditional on $N(t) = n$, the distribution function is given by (1.14) according to section 1.2.3. In order to know the price of stock under model (2.4), we need to study the convolution of Brownian motion and sums of exponential random variable. Denote the density of $rt + \sigma_{\epsilon_t} B_t + \sum_{i=1}^{N(t)} \xi_i$ as $f_{Z_t + \sum_{i=1}^{N(t)} \xi_i}(s)$, where ξ_i is the exponential random variable and $Z_t = rt + \sigma_{\epsilon_t} B_t$. From independence we have the p.d.f.:

$$
\begin{aligned}
f_{Z_t + \sum_{i=1}^{N(t)} \xi_i}(s) &= \int_{-\infty}^{+\infty} f_{\sum_{i=1}^{N(t)} \xi_i}(s-x) f_{Z_t}(x) dx \\
&= e^{-s\theta}\theta^n \int_{-\infty}^{s} \frac{e^{x\theta}(s-x)^{n-1}}{(n-1)!} \frac{1}{\sigma_{\epsilon_t}\sqrt{2\pi t}} e^{(x-rt-x_0)^2/2\sigma_{\epsilon_t}^2 t} dx. \quad (2.20)
\end{aligned}
$$

In S_t space (i.e. for GBM), conditional on $S_t \geq K$, where K is the strike price, then $f_{Z_t+\sum_{i=1}^{N(t)}\xi_i}(s)$ becomes

$$\begin{aligned} f_{Z_t+\sum_{i=1}^{N(t)}\xi_i}(s) &= \int_0^{+\infty} \frac{1}{y+K} f_{\sum_{i=1}^{N(t)}\xi_i}(s-\ln(y+K)) f_{Z_t}(\ln(y+K))dy \\ &= e^{-s\theta}\theta^n \int_0^s \frac{e^{\ln(y+K)\theta}(s-\ln(y+K))^{n-1}}{(y+K)(n-1)!} \\ &\quad \phi(\ln(y+K), m(t), v(t))dy \qquad (2.21) \end{aligned}$$

where the dummy variable $s = \ln(S_t)$ and

$$m(t) = \ln S_0 + \frac{1}{2}(\sigma_1^2 - \sigma_0^2)t + (r - \frac{1}{2}\sigma_1^2)T, \qquad (2.22)$$

$$v(t) = (\sigma_0^2 - \sigma_1^2)t + \sigma_1^2 T. \qquad (2.23)$$

Thus, the European call option value conditional on $N(t) = n$ and given current economic state i is given by

$$\begin{aligned} C_n(.,i) &= e^{-rT}(\theta^n) \int_0^T \int_0^\infty \int_0^s \frac{ye^{(\ln(y+K)-s)\theta}(s-\ln(y+K))^{n-1}}{(y+K)(n-1)!} \\ &\quad \phi(\ln(y+K), m(t), v(t)) f_i(t,T)dydsdt. \qquad (2.24) \end{aligned}$$

If we continue to assume the initial probability of state i is $\pi(i)$, the option price becomes

$$C = \sum_{i=0}^{1}\sum_{n=0}^{\infty} \pi(i) C_n(\cdot, i) \frac{(\lambda T)^n e^{-\lambda T}}{n!}. \qquad (2.25)$$

2.4.3 Constant Percentage of Jump Size with Compensation

Consider the following model:

$$\frac{dS_t}{S_{t-}} = (r-c)dt + \sigma_{\epsilon_t} dB_t + d(\sum_{i=1}^{N(t)}(K^* - 1))$$

where $c = \frac{E[N(t)]}{t}K^* = \rho K^*$, K^* is constant jump size and ρ is the jump intensity. This model is the modified Merton jump diffusion model, which has the additional Markov switching component in the model. Obviously from the discussion above, we have

$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma_{\epsilon_t}^2 - c)t + \sigma_{\epsilon_t} B_t + K^* N(t)\}.$$

The key is to find the convolution of a normal random variable and the compound Poisson variable, which is $Z_t + \sum_{i=0}^{N(t)} \ln(K^*)$. Given current state i and $N(t) = n$, the European call option value is then

$$\begin{aligned} C_n(\cdot, i) &= e^{-rT} \int_0^\infty \int_0^T \frac{y}{y+K} \phi(ln(y+K), m(t), v(t)) f_i(t,T) dt dy \\ &= \int_0^T \left(e^{-rT + \frac{v(t)}{2} + m(t)} \Phi(\frac{m(t) + v(t) - \ln K}{\sqrt{v(t)}}) - e^{-rT} K \Phi(\frac{m(t) - \ln K}{\sqrt{v(t)}}) \right) \\ &\times \; f_i(t,T) dt \end{aligned} \tag{2.26}$$

where $m(t) = lnS_0 + n\ln(K^*) - cT + \frac{1}{2}(\sigma_1^2 - \sigma_0^2)t + (r - \frac{1}{2}\sigma_1^2)T$, $v(t) = (\sigma_0^2 - \sigma_1^2)t + \sigma_1^2 T$. If we continue to assume the initial probability of state i is $\pi(i)$, the option value becomes

$$C = \sum_{i=0}^{1} \sum_{n=0}^{\infty} \pi(i) C_n(\cdot, i) \frac{(\lambda T)^n e^{-\lambda T}}{n!}. \tag{2.27}$$

The formulation for option pricing is exactly the same as the case of lognormal percentage of change.

2.5 Transition Probability and Path Integral Approach

From the previous section we have the Markov chain transition matrix given by equation (1.17). Therefore,

$$p_{00}(t) = \frac{\lambda_1 + \lambda_0 e^{-(\lambda_0+\lambda_1)t}}{\lambda_0 + \lambda_1}; \quad p_{01}(t) = \frac{\lambda_0 - \lambda_0 e^{-(\lambda_0+\lambda_1)t}}{\lambda_0 + \lambda_1}$$

and

$$p_{10}(t) = \frac{\lambda_1 - \lambda_1 e^{-(\lambda_0+\lambda_1)t}}{\lambda_0 + \lambda_1}; \quad p_{11}(t) = \frac{\lambda_0 + \lambda_1 e^{-(\lambda_0+\lambda_1)t}}{\lambda_0 + \lambda_1}.$$

In this case, suppose the process can only switch once from one regime to another during some small time period δt_i. We evenly partition the time interval $[0, t]$ into $\bigcup_{i=1}^{N}[t_{i-1}, t_i]$, where $t_0 = 0$ and $t_N = t$. Generally, δt_i is not constant, i.e, unevenly partitioned, but the following discussion just focuses on the case of constant $\delta t_i = \delta t$. Hence the corresponding discrete Markov transition matrix after i time steps is

$$P^{(i)}(t) = \frac{1}{\lambda_0 + \lambda_1} \begin{pmatrix} \lambda_1 + \lambda_0 e^{-(\lambda_0+\lambda_1)i\delta t} & \lambda_0 - \lambda_0 e^{-(\lambda_0+\lambda_1)i\delta t} \\ \lambda_1 - \lambda_1 e^{-(\lambda_0+\lambda_1)i\delta t} & \lambda_0 + \lambda_1 e^{-(\lambda_0+\lambda_1)i\delta t} \end{pmatrix}. \tag{2.28}$$

In particular, the transition matrix for one step is:

$$P^{(1)}(t) = \frac{1}{\lambda_0 + \lambda_1} \begin{pmatrix} \lambda_1 + \lambda_0 e^{-(\lambda_0+\lambda_1)\delta t} & \lambda_0 - \lambda_0 e^{-(\lambda_0+\lambda_1)\delta t} \\ \lambda_1 - \lambda_1 e^{-(\lambda_0+\lambda_1)\delta t} & \lambda_0 + \lambda_1 e^{-(\lambda_0+\lambda_1)\delta t} \end{pmatrix}. \tag{2.29}$$

Assume the initial state probability for state 0 and state 1 is π_0 and π_1, respectively, so that the probability that the process is in state 0 and 1 after $k-1$ steps of switching is given, respectively, by:

$$\begin{aligned} p_0^{(k-1)} &= \frac{1}{\lambda_0 + \lambda_1}\left(\pi_0(\lambda_1 + \lambda_0 e^{-(\lambda_0+\lambda_1)(k-1)\delta t}) + \pi_1(\lambda_1 - \lambda_1 e^{-(\lambda_0+\lambda_1)(k-1)\delta t})\right) \\ &= \frac{1}{\lambda_0 + \lambda_1}\left(\lambda_1 + \pi_0\lambda_0 e^{-(\lambda_0+\lambda_1)(k-1)\delta t} - \pi_1\lambda_1 e^{-(\lambda_0+\lambda_1)(k-1)\delta t}\right) \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} p_1^{(k-1)} &= \frac{1}{\lambda_0 + \lambda_1}\left(\pi_0(\lambda_0 - \lambda_0 e^{-(\lambda_0+\lambda_1)(k-1)\delta t}) + \pi_1(\lambda_0 + \lambda_1 e^{-(\lambda_0+\lambda_1)(k-1)\delta t})\right) \\ &= \frac{1}{\lambda_0 + \lambda_1}\left(\lambda_0 - \pi_0\lambda_0 e^{-(\lambda_0+\lambda_1)(k-1)\delta t} + \pi_1\lambda_1 e^{-(\lambda_0+\lambda_1)(k-1)\delta t}\right). \end{aligned} \tag{2.31}$$

From the expressions we used $\pi_0 + \pi_1 = 1$ in (2.30) and (2.31), we also have the following equality:

$$p_0^{(k-1)} + p_1^{(k-1)} = 1, \;\; k \geq 1. \tag{2.32}$$

2.5.1 X_t - space Markov Switching Transition Probability Density with No Absorption

In this section, we consider the transition probability density using a path integral approach. Consider a finite time transition from x_{k-1} to x_k in time δt,

$$\begin{aligned} u_0(x_k, x_{k-1}, \delta t) &= (p_1^{(k-1)} p_{10} + p_0^{(k-1)} p_{00}) u_0^{(0)}(x_k, x_{k-1}, \delta t) \\ &+ (p_1^{(k-1)} p_{11} + p_0^{(k-1)} p_{01}) u_0^{(1)}(x_k, x_{k-1}, \delta t) \qquad (2.33) \end{aligned}$$

where p_{ij} is the transition probability from state i to state j, $p_i^{(k-1)}$ is the probability that the process is in state i after $k-1$ steps of switching. Note: we use subscript 0 to denote the densities for the case of no absorption of path, where $u_0^{(0)}$ and $u_0^{(1)}$ are the respective transition p.d.f. for state 0 and 1 with no absorption (see Equations (2.53 and 2.54)). Hence, the joint transition density for x_t process to start at x_0 and end at $x_N = x$ after time $N\delta t$ is

$$\begin{aligned} u_0(x, x_0, N\delta t) &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} u_0(x_i, x_{i-1}, \delta t) dx_{N-1} \cdots dx_1 \\ &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} \{(p_1^{(i-1)} p_{10} + p_0^{(i-1)} p_{00}) u_0^{(0)}(x_i, x_{i-1}, \delta t) \\ &+ (p_1^{(i-1)} p_{11} + p_0^{(i-1)} p_{01}) u_0^{(1)}(x_i, x_{i-1}, \delta t)\} dx_{N-1} \cdots dx_1 \qquad (2.34) \end{aligned}$$

Let $p_i = p_1^{(i-1)} p_{10} + p_0^{(i-1)} p_{00} \equiv p_0^{(i)}$ and $q_i = p_1^{(i-1)} p_{11} + p_0^{(i-1)} p_{01} \equiv p_1^{(i)}$, where $p_0^{(i)}$ and $p_1^{(i)}$ are defined in (2.30) and (2.31) respectively. Then, the above path integral can be simplified by the use of Chapman-Kolmogorov identity as follows,

$$
\begin{aligned}
u_0(x, x_0, N\delta t) &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} (p_i u_0^{(0)}(x_i, x_{i-1}, \delta t) + q_i u_0^{(1)}(x_i, x_{i-1}, \delta t)) dx_{N-1} \cdots dx_1 \\
&= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \Bigg(\prod_{i=1}^{N} p_i u_0^{(0)}(x_i, x_{i-1}, \delta t) + \prod_{i=1}^{N} q_i u_0^{(1)}(x_i, x_{i-1}, \delta t) \\
&+ \sum_{\omega(\mathbf{k})} \prod_{i=1}^{N} m^{(k_i)}(p_i, q_i) \times \prod_{i=1}^{N} m^{(k_i)}(u_0^{(0)}(x_i, x_{i-1}, \delta t), u_0^{(1)}(x_i, x_{i-1}, \delta t)) \Bigg) \\
& dx_{N-1} \cdots dx_1 \\
&= \prod_{i=1}^{N} p_i u_0^{(0)}(x, x_0, N\delta t) + \prod_{i=1}^{N} q_i u_0^{(1)}(x, x_0, N\delta t) \\
&+ \sum_{\omega(\mathbf{k})} \prod_{i=1}^{N} m^{(k_i)}(p_i, q_i) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, N\delta t) \\
&= \prod_{i=1}^{N} p_i u_0^{(0)}(x, x_0, N\delta t) + \prod_{i=1}^{N} q_i u_0^{(1)}(x, x_0, N\delta t) \\
&+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, N\delta t)
\end{aligned}
\tag{2.35}
$$

where $c(\omega(\mathbf{k})) = \prod_{i=1}^{N} m^{(k_i)}(p_i, q_i)$ and $\mathbf{k} = (k_1, k_2, \cdots, k_N)$, e.g., $\mathbf{k} = (0, 1, 0, 1, 1, \cdots)$ is a sequence of 0,1 and $m^{(k_i)}$ is a chooser function defined by the following form:

$$
m^{(k_i)}(a, b) \equiv \begin{cases} a, & k_i{=}0; \\ b, & k_i{=}1. \end{cases} \tag{2.36}
$$

$\omega(\mathbf{k})$={any possible combination of the vector (binary) sequence $\mathbf{k}$}. In particular, for standard Brownian motion:

$$
u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, N\delta t) = \sqrt{\frac{b}{\pi}} \exp\left\{-b(x - x_0)^2\right\}, \tag{2.37}
$$

$$
b = \frac{\prod_{i=1}^{N} a_i^{(k_i)}}{\sum_{i=1}^{N} \prod_{j=1, i \neq j} a_j^{(k_j)}}, \tag{2.38}
$$

$$
a_i^{(k_i)} = \frac{1}{2\sigma_i^2(t_i)}, \tag{2.39}
$$

and

$$\sigma_i \equiv \sigma^{(k_i)}(t_i). \tag{2.40}$$

In our case,

$$b = \frac{1}{2N\delta t \cdot \frac{1}{N}\sum_{j=1}^{N}\sigma^2(t_j)},$$

as will be seen in the next section. By replacing $N\delta t$ by t, we have the transition probability density $u_0(x, x_0, t)$ for this Markov switching Brownian motion process in closed-form as

$$\begin{aligned} u_0(x, x_0, t) &= \prod_{i=1}^{N} p_i u_0^{(0)}(x, x_0, t) + \prod_{i=1}^{N} q_i u_0^{(1)}(x, x_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t). \end{aligned} \tag{2.41}$$

Defining

$$\prod_{i=1}^{N} p_i = \alpha_N; \tag{2.42}$$

$$\prod_{i=1}^{N} q_i = \beta_N \tag{2.43}$$

then (2.41) becomes:

$$\begin{aligned} u_0(x, x_0, t) &= \alpha_N u_0^{(0)}(x, x_0, t) + \beta_N u_0^{(1)}(x, x_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t). \end{aligned} \tag{2.44}$$

2.5.2 Effective Volatility for One Switching Sequence

Consider the transition density for a Markov switching diffusion process with a particular switching sequence $\omega(\mathbf{k}) = (k_1, k_2, \cdots, k_N)$. In this case the path integrals:

$$u_0^{(\omega(\mathbf{k}))}(x, x_0, N\delta t) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} u_0^{(k_i)}(x_i, x_{i-1}, \delta t) dx_{N-1} \cdots dx_1$$

where $u_0^{(k_i)}(x_i, x_{i-1}, \delta t) = \sqrt{\frac{a_i}{\pi}} \exp\left\{-a_i(x_i - x_{i-1})^2\right\}$. Using the following identity

$$\int_{-\infty}^{+\infty} \exp\left\{-a(x-q)^2 - c(p-x)^2\right\} dx = \sqrt{\frac{\pi}{a+c}} \exp\left\{-\frac{ac}{a+c}(p-q)^2\right\}, \quad (2.45)$$

we can prove that $u_0(x, x_0, N\delta t)$ is a Gaussian density, as follows:

$$\begin{aligned}
u_0^{(\omega(\mathbf{k}))}(x, x_0, N\delta t) &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} u_0(x_i, x_{i-1}, \delta t) dx_{N-1} \cdots dx_1 \\
&= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-1} \prod_{i=1}^{N} \sqrt{\frac{a_i}{\pi}} \exp\left\{-a_i(x_i - x_{i-1})^2\right\} dx_{N-1} \cdots dx_1 \\
&= \left(\prod_{i=1}^{N} \sqrt{\frac{a_i}{\pi}}\right) \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-2} \prod_{i=3}^{N} \exp\left\{-a_i(x_i - x_{i-1})^2\right\} \\
&\times \int_{-\infty}^{+\infty} \exp\{-a_2(x_2 - x_1)^2 - a_1(x_1 - x_0)^2\} dx_{N-1} \cdots dx_2 dx_1 \\
&= \left(\prod_{i=1}^{N} \sqrt{\frac{a_i}{\pi}}\right) \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-2} \prod_{i=3}^{N} \exp\left\{-a_i(x_i - x_{i-1})^2\right\} dx_{N-1} \cdots dx_2 \\
&\times \sqrt{\frac{\pi}{a_1 + a_2}} \exp\left\{-\frac{a_1 a_2}{a_1 + a_2}(x_2 - x_0)^2\right\} \\
&= \left(\prod_{i=1}^{N} \sqrt{\frac{a_i}{\pi}}\right) \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-3} \prod_{i=4}^{N} \exp\left\{-a_i(x_i - x_{i-1})^2\right\} \\
&\times \sqrt{\frac{\pi}{a_1 + a_2}} \int_{-\infty}^{+\infty} \exp\{-\frac{a_1 a_2}{a_1 + a_2}(x_2 - x_0)^2 - a_3(x_3 - x_2)^2\} dx_{N-1} \cdots dx_3 dx_2 \\
&= \left(\prod_{i=1}^{N} \sqrt{\frac{a_i}{\pi}}\right) \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N-3} \prod_{i=4}^{N} \exp\left\{-a_i(x_i - x_{i-1})^2\right\} dx_{N-1} \cdots dx_3 \\
&\times (\sqrt{\pi})^2 \sqrt{\frac{1}{a_1 a_2 + a_2 a_3 + a_1 a_3}} \exp\left\{-\frac{a_1 a_2 a_3}{a_1 a_2 + a_2 a_3 + a_1 a_3}(x_3 - x_0)^2\right\}. (2.46)
\end{aligned}$$

By mathematical induction, we then have

$$
\begin{aligned}
u_0^{(\omega(\mathbf{k}))}(x, x_0, t) &= u_0^{(\omega(\mathbf{k}))}(x, x_0, N\delta t) = \sqrt{\frac{\prod_{i=1}^N a_i}{\pi \sum_{i=1}^N \prod_{j=1, i\neq j}^N a_j}} \\
&\times \exp\left\{ -\frac{\prod_{i=1}^N a_i}{\sum_{i=1}^N \prod_{j=1, i\neq j}^N a_j} (x - x_0)^2 \right\}. \qquad (2.47)
\end{aligned}
$$

This is obviously a Gaussian density. We define $a_i = \frac{1}{2\sigma(t_i)^2 \delta t}$, where $\sigma(t_i) = \sigma_0$ if the process is in state i at time t_i, otherwise it is equal to σ_1 at time t_i. Therefore,

$$
\begin{aligned}
\frac{\prod_{i=1}^N a_i}{\sum_{i=1}^N \prod_{j=1, i\neq j}^N a_j} &= \frac{1}{\sum_{j=1}^N \frac{1}{a_j}} = \frac{1}{\sum_{j=1}^N 2\sigma^2(t_j)\delta t} = \frac{1}{2N\delta t \frac{1}{N} \sum_{j=1}^N \sigma^2(t_j)} \\
&= \frac{1}{2t \frac{1}{N} \sum_{j=1}^N \sigma^2(t_j)}.
\end{aligned}
$$

Let $\bar{\sigma}(t) = \sqrt{\frac{1}{N} \sum_{j=1}^N \sigma^2(t_j)}$, this is the effective volatility for a single path. As $\delta t \to 0$, $\bar{\sigma}(t) = \sqrt{\frac{1}{t} \int_0^t \sigma^2(s) ds}$, assuming $\sigma^2(s)$ is integrable.

2.5.3 Markov Switching Transition Probability Density with Absorption

In this section, we determine the transition probability density with a single absorbing (killing) barrier level x_H for Brownian motion with two-state Markov switching process. From the result of the previous sections, the transition density for the free boundary Markov switching process is given by

$$
\begin{aligned}
u_0(x, x_0, t) &= \alpha_N u_0^{(0)}(x, x_0, t) + \beta_N u_0^{(1)}(x, x_0, t) \\
&+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t). \qquad (2.48)
\end{aligned}
$$

By applying the reflection principle, we have the new transition density denoted by $u(x_H, x, x_0, t)$ for the two-state Markov switching Brownian motion with barrier

x_H:

$$
\begin{aligned}
u(x_H, x, x_0, t) &= u_0(x, x_0, t) - u_0(x, 2x_H - x_0, t) \\
&= \alpha_N u_0^{(0)}(x, x_0, t) + \beta_N u_0^{(1)}(x, x_0, t) + \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t) \\
&- \alpha_N u_0^{(0)}(x, 2x_H - x_0, t) - \beta_N u_0^{(1)}(x, 2x_H - x_0, t) \\
&- \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u_0^{(\{\omega(\mathbf{k})\})}(x, 2x_H - x_0, t) \\
&= \alpha_N \left(u_0^{(0)}(x, x_0, t) - u_0^{(0)}(x, 2x_H - x_0, t) \right) \\
&+ \beta_N \left(u_0^{(1)}(x, x_0, t) - u_0^{(1)}(x, 2x_H - x_0, t) \right) \\
&+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \left(u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t) - u_0^{(\{\omega(\mathbf{k})\})}(x, 2x_H - x_0, t) \right) \\
&= \alpha_N u^{(0)}(x_H, x, x_0, t) + \beta_N u^{(1)}(x_H, x, x_0, t) \\
&+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) u^{(\{\omega(\mathbf{k})\})}(x_H, x, x_0, t), \qquad (2.49)
\end{aligned}
$$

where $u^{(i)}(x_H, x, x_0, t)$ are the transition densities for Brownian motion with an absorbing barrier at x_H assuming a given state i (see (2.62) to (2.64)). The transition density for Brownian motion with drift r, and with volatility σ and no barrier is

$$u_0(x, x_0, t) = \frac{e^{-(x-x_0-rt)^2/2\sigma^2 t}}{\sigma\sqrt{2\pi t}}. \qquad (2.50)$$

Rewriting gives

$$u_0(x, x_0, t) = e^{\frac{r}{\sigma^2}(x-x_0) - \frac{r^2}{2\sigma^2}t} g_0(x, x_0, t) \qquad (2.51)$$

where

$$g_0(x, x_0, t) = \frac{e^{-(x-x_0)^2/2\sigma^2 t}}{\sigma\sqrt{2\pi t}}. \qquad (2.52)$$

In our case, in the regime (state 0) of volatility σ_0

$$u_0^{(0)}(x, x_0, t) = \frac{e^{-(x-x_0-rt)^2/2\sigma_0^2 t}}{\sigma_0\sqrt{2\pi t}} \qquad (2.53)$$

and in the regime (state 1) of volatility σ_1 gives

$$u_0^{(1)}(x, x_0, t) = \frac{e^{-(x-x_0-rt)^2/2\sigma_1^2 t}}{\sigma_1\sqrt{2\pi t}}. \tag{2.54}$$

More generally, the path integral for one given switching sequence $\mathbf{k} = (k_1, k_2, \cdots, k_N)$ is given by

$$u_0^{(\omega(\mathbf{k}))}(x, x_0, t) = \frac{e^{-(x-x_0-rt)^2/2\bar{\sigma}^2 t}}{\bar{\sigma}\sqrt{2\pi t}} \tag{2.55}$$

where $\bar{\sigma}^2 = \frac{1}{N}\sum_{j=1}^N \sigma^2(t_j)$, $\sigma(t_j) = \sigma_{k_j}$. Rewriting all the densities above gives

$$u_0^{(0)}(x, x_0, t) = e^{\frac{r}{\sigma_0^2}(x-x_0)-\frac{r^2}{2\sigma_0^2}t} g_0^{(0)}(x, x_0, t), \tag{2.56}$$

$$u_0^{(1)}(x, x_0, t) = e^{\frac{r}{\sigma_1^2}(x-x_0)-\frac{r^2}{2\sigma_1^2}t} g_0^{(1)}(x, x_0, t), \tag{2.57}$$

$$u_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t) = e^{\frac{r}{\bar{\sigma}^2}(x-x_0)-\frac{r^2}{2\bar{\sigma}^2}t} g_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t), \tag{2.58}$$

where

$$g_0^{(0)}(x, x_0, t) = \frac{e^{-(x-x_0)^2/2\sigma_0^2 t}}{\sigma_0\sqrt{2\pi t}}, \tag{2.59}$$

$$g_0^{(1)}(x, x_0, t) = \frac{e^{-(x-x_0)^2/2\sigma_1^2 t}}{\sigma_1\sqrt{2\pi t}}, \tag{2.60}$$

$$g_0^{(\{\omega(\mathbf{k})\})}(x, x_0, t) = \frac{e^{-(x-x_0)^2/2\bar{\sigma}^2 t}}{\bar{\sigma}\sqrt{2\pi t}}. \tag{2.61}$$

Therefore, the transition densities for paths with absorbing barrier at x_H are

$$u^{(0)}(x_H, x, x_0, t) = \frac{e^{\frac{r}{\sigma_0^2}(x-x_0)-\frac{r^2}{2\sigma_0^2}t}}{\sigma_0\sqrt{2\pi t}}\left(e^{-(x-x_0)^2/2\sigma_0^2 t} - e^{-(x+x_0-2x_H)^2/2\sigma_0^2 t}\right), \tag{2.62}$$

$$u^{(1)}(x_H, x, x_0, t) = \frac{e^{\frac{r}{\sigma_1^2}(x-x_0)-\frac{r^2}{2\sigma_1^2}t}}{\sigma_1\sqrt{2\pi t}}\left(e^{-(x-x_0)^2/2\sigma_1^2 t} - e^{-(x+x_0-2x_H)^2/2\sigma_1^2 t}\right), \tag{2.63}$$

and

$$u^{(\{\omega(\mathbf{k})\})}(x_H, x, x_0, t) = \frac{e^{\frac{r}{\bar{\sigma}^2}(x-x_0)-\frac{r^2}{2\bar{\sigma}^2}t}}{\bar{\sigma}\sqrt{2\pi t}}\left(e^{-(x-x_0)^2/2\bar{\sigma}^2 t} - e^{-(x+x_0-2x_H)^2/2\bar{\sigma}^2 t}\right). \tag{2.64}$$

2.6 First Passage Times for Geometric Brownian Motion with Drift and Markov Switching

2.6.1 Closed-form Transition Probability for Markov Switching

In this section, we move from x_t-space to S_t-space, i.e., from simple Brownian motion to geometric Brownian motion. Consider $S_t > 0$ obeying the SDE for GBM:

$$\frac{dS_t}{S_t} = rdt + \sigma_{\epsilon_t} dB_t. \tag{2.65}$$

By defining the variable transformation $x_t = X(S_t) = \log(S_t)$, then the process x_t has SDE

$$dx_t = (r - \frac{\sigma_{\epsilon_t}^2}{2})dt + \sigma_{\epsilon_t} dB_t. \tag{2.66}$$

Changing variable with Jacobian $dX/dS = 1/S$ gives a relationship between the S_t-space and x_t-space densities:

$$U(S, S_0, t) = \frac{1}{S} u_{r-\frac{1}{2}\sigma_{\epsilon_t}^2}(X(S), X(S_0), t). \tag{2.67}$$

Note: The subscript in u is used to denote the density for the x_t process where the drift r is replaced by $r - \frac{1}{2}\sigma_{\epsilon_t}^2$. Using this general relationship and the above results for the u densities, we have the following results for the respective cases of S_t-space free (no barrier) motion:

$$\begin{aligned} U_0^{(0)}(S, S_0, t) &= \frac{1}{S} u_0^{(0)}(\log S, \log S_0, t) \\ &= \frac{1}{\sigma_0 S \sqrt{2\pi t}} e^{-[\log S - \log S_0 - (r - \sigma_0^2/2)t]^2 / 2\sigma_0^2 t}, \end{aligned} \tag{2.68}$$

$$\begin{aligned} U_0^{(1)}(S, S_0, t) &= \frac{1}{S} u_0^{(1)}(\log S, \log S_0, t) \\ &= \frac{1}{\sigma_1 S \sqrt{2\pi t}} e^{-[\log S - \log S_0 - (r - \sigma_1^2/2)t]^2 / 2\sigma_1^2 t}, \end{aligned} \tag{2.69}$$

and

$$U_0^{(\{\omega(\mathbf{k})\})}(S,S_0,t) = \frac{1}{S}u_0^{(\{\omega(\mathbf{k})\})}(\log S,\log S_0,t)$$
$$= \frac{1}{\bar{\sigma}S\sqrt{2\pi t}}e^{-[\log S-\log S_0-(r-\bar{\sigma}^2/2)t]^2/2\bar{\sigma}^2 t}. \tag{2.70}$$

These expressions are the familiar lognormal densities for the Black-Scholes model with respective volatilities. From these formulas, the transition density for two-state Markov switching geometric Brownian motion with no barrier is

$$U_0(S,S_0,t) = \alpha_N U_0^{(0)}(S,S_0,t) + \beta_N U_0^{(1)}(S,S_0,t)$$
$$+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k}))U_0^{(\{\omega(\mathbf{k})\})}(S,S_0,t). \tag{2.71}$$

To obtain the density for the case of a single barrier at $S = H$, the barrier point in the two spaces are related by $x_H = X(H) = \log H$. By simply substituting the single-barrier densities for the x_t process into (2.67), we obtain the analogous forms to (2.62), (2.63), and (2.64):

$$U^{(0)}(H,S,S_0,t) = \frac{1}{S}u^{(0)}(\log H,\log S,\log S_0,t)$$
$$= \frac{1}{\sigma_0 S\sqrt{2\pi t}}e^{\frac{r-\frac{1}{2}\sigma_0^2}{\sigma_0^2}(\log S-\log S_0)-\frac{(r-\frac{1}{2}\sigma_0^2)^2}{2\sigma_0^2}t}$$
$$\times \left(e^{-(\log S-\log S_0)^2/2\sigma_0^2 t} - e^{-(\log S+\log S_0-2\log H)^2/2\sigma_0^2 t}\right), \tag{2.72}$$

$$U^{(1)}(H,S,S_0,t) = \frac{1}{S}u^{(1)}(\log H,\log S,\log S_0,t)$$
$$= \frac{1}{\sigma_1 S\sqrt{2\pi t}}e^{\frac{r-\frac{1}{2}\sigma_1^2}{\sigma_1^2}(\log S-\log S_0)-\frac{(r-\frac{1}{2}\sigma_1^2)^2}{2\sigma_1^2}t}$$
$$\times \left(e^{-(\log S-\log S_0)^2/2\sigma_1^2 t} - e^{-(\log S+\log S_0-2\log H)^2/2\sigma_1^2 t}\right), \tag{2.73}$$

$$U^{(\{\omega(\mathbf{k})\})}(H,S,S_0,t) = \frac{1}{S}u^{(\{\omega(\mathbf{k})\})}(\log H,\log S,\log S_0,t)$$
$$= \frac{1}{\bar{\sigma} S\sqrt{2\pi t}}e^{\frac{r-\frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}^2}(\log S-\log S_0)-\frac{(r-\frac{1}{2}\bar{\sigma}^2)^2}{2\bar{\sigma}^2}t}$$
$$\times \left(e^{-(\log S-\log S_0)^2/2\bar{\sigma}^2 t} - e^{-(\log S+\log S_0-2\log H)^2/2\bar{\sigma}^2 t}\right) \tag{2.74}$$

Therefore, the transition density for two-state Markov switching geometric Brownian motion with a single barrier at $S = H$ is:

$$\begin{aligned} U(H, S, S_0, t) &= \alpha_N U^{(0)}(H, S, S_0, t) + \beta_N U^{(1)}(H, S, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) U^{(\{\omega(\mathbf{k})\})}(H, S, S_0, t). \end{aligned} \tag{2.75}$$

From this result the probability that the asset price process starting at $S_0 \leq H$ will attain the *upper* barrier H and will be absorbed in time t is:

$$\begin{aligned} \Phi(H, S_0, t) &= 1 - \int_0^H U(H, S, S_0, t) dS \\ &= 1 - \alpha_N \int_0^H U^{(0)}(H, S, S_0, t) dS - \beta_N \int_0^H U^{(1)}(H, S, S_0, t) dS \\ &- \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \int_0^H U^{(\{\omega(\mathbf{k})\})}(H, S, S_0, t) dS \\ &= \alpha_N \Phi^{(0)}(H, S_0, t) + \beta_N \Phi^{(1)}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t). \end{aligned} \tag{2.76}$$

Similarly, the probability that the asset price process starting at $S_0 \geq H$ attains the *lower* barrier H and is absorbed in time t is:

$$\begin{aligned} \Phi(H, S_0, t) &= 1 - \int_H^\infty U(H, S, S_0, t) dS \\ &= 1 - \alpha_N \int_H^\infty U^{(0)}(H, S, S_0, t) dS - \beta_N \int_H^\infty U^{(1)}(H, S, S_0, t) dS \\ &- \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \int_H^\infty U^{(\{\omega(\mathbf{k})\})}(H, S, S_0, t) dS \\ &= \alpha_N \Phi^{(0)}(H, S_0, t) + \beta_N \Phi^{(1)}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t). \end{aligned} \tag{2.77}$$

In equation (2.76) and (2.77) we used the identity

$$\alpha_N + \beta_N + \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) = 1 \tag{2.78}$$

which follows from (2.32). For $S_0 \geq H$:

$$\begin{aligned}\Phi^{(0)}(H,S_0,t) &= 1-N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_0^2)t}{\sigma_0\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma_0^2}-1} N\left(\frac{-\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_0^2)t}{\sigma_0\sqrt{t}}\right), \end{aligned}\tag{2.79}$$

$$\begin{aligned}\Phi^{(1)}(H,S_0,t) &= 1-N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_1^2)t}{\sigma_1\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma_1^2}-1} N\left(\frac{-\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_1^2)t}{\sigma_1\sqrt{t}}\right), \end{aligned}\tag{2.80}$$

and

$$\begin{aligned}\Phi^{(\{\omega(\mathbf{k})\})}(H,S_0,t) &= 1-N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\bar{\sigma}^2)t}{\bar{\sigma}\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\bar{\sigma}^2}-1} N\left(\frac{-\log\frac{S_0}{H}+(r-\frac{1}{2}\bar{\sigma}^2)t}{\bar{\sigma}\sqrt{t}}\right). \end{aligned}\tag{2.81}$$

Here we denote the cumulative standard normal function by $N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$.

For $S_0 \leq H$:

$$\begin{aligned}\Phi^{(0)}(H,S_0,t) &= N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_0^2)t}{\sigma_0\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma_0^2}-1} N\left(\frac{\log\frac{S_0}{H}-(r-\frac{1}{2}\sigma_0^2)t}{\sigma_0\sqrt{t}}\right), \end{aligned}\tag{2.82}$$

$$\begin{aligned}\Phi^{(1)}(H,S_0,t) &= N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\sigma_1^2)t}{\sigma_1\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma_1^2}-1} N\left(\frac{\log\frac{S_0}{H}-(r-\frac{1}{2}\sigma_1^2)t}{\sigma_1\sqrt{t}}\right), \end{aligned}\tag{2.83}$$

and

$$\begin{aligned}\Phi^{(\{\omega(\mathbf{k})\})}(H,S_0,t) &= N\left(\frac{\log\frac{S_0}{H}+(r-\frac{1}{2}\bar{\sigma}^2)t}{\bar{\sigma}\sqrt{t}}\right)\\ &+ \left(\frac{H}{S_0}\right)^{\frac{2r}{\bar{\sigma}^2}-1} N\left(\frac{\log\frac{S_0}{H}-(r-\frac{1}{2}\bar{\sigma}^2)t}{\bar{\sigma}\sqrt{t}}\right). \end{aligned}\tag{2.84}$$

We should realize that

$$\Phi(S_0, S_0, t) = 1 \tag{2.85}$$

from the fact $\Phi^{(0)}(S_0, S_0, t)$, $\Phi^{(1)}(S_0, S_0, t)$, $\Phi^{(\{\omega(\mathbf{k})}(S_0, S_0, t)$ are all equal to 1, as required by the physical boundary condition for paths that begin at the absorbing barrier. Since $\Phi(H, S_0, t)$ is a cumulative function, the probability density f for the first passage time $\tau_H = t$ for hitting the barrier must be given by differentiation:

$$f(H, S_0, t) = \frac{\partial \Phi(H, S_0, t)}{\partial t}. \tag{2.86}$$

Therefore, from(2.76) or (2.77), we have the first passage time density

$$\begin{aligned} f(H, S_0, t) &= \frac{\partial \alpha_N}{\partial t} \Phi^{(0)}(H, S_0, t) + \alpha_N \frac{\partial \Phi^{(0)}(H, S_0, t)}{\partial t} + \frac{\partial \beta_N}{\partial t} \Phi^{(1)}(H, S_0, t) \\ &+ \beta_N \frac{\partial \Phi^{(1)}(H, S_0, t)}{\partial t} + \sum_{\omega(\mathbf{k})} \frac{\partial c(\omega(\mathbf{k}))}{\partial t} \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) \frac{\partial \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t)}{\partial t}. \end{aligned} \tag{2.87}$$

Using the expression in (2.86), we have

$$\begin{aligned} f(H, S_0, t) &= \frac{\partial \alpha_N}{\partial t} \Phi^{(0)}(H, S_0, t) + \alpha_N f^{(0)}(H, S_0, t) + \frac{\partial \beta_N}{\partial t} \Phi^{(1)}(H, S_0, t) \\ &+ \beta_N f^{(1)}(H, S_0, t) + \sum_{\omega(\mathbf{k})} \frac{\partial c(\omega(\mathbf{k}))}{\partial t} \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) f^{(\{\omega(\mathbf{k})\})}(H, S_0, t), \end{aligned} \tag{2.88}$$

where

$$f^{(0)}(H, S_0, t) = \frac{|\log(S_0/H)|}{\sigma_0 t^3 \sqrt{2\pi}} e^{-\left[\log(S_0/H) + (r - \frac{1}{2}\sigma_0^2)t\right]^2 / 2\sigma_0^2 t}, \tag{2.89}$$

$$f^{(1)}(H, S_0, t) = \frac{|\log(S_0/H)|}{\sigma_1 t^3 \sqrt{2\pi}} e^{-\left[\log(S_0/H) + (r - \frac{1}{2}\sigma_1^2)t\right]^2 / 2\sigma_1^2 t}, \tag{2.90}$$

$$f^{(\{\omega(\mathbf{k})\})}(H, S_0, t) = \frac{|\log(S_0/H)|}{\bar{\sigma} t^3 \sqrt{2\pi}} e^{-\left[\log(S_0/H) + (r - \frac{1}{2}\bar{\sigma}^2)t\right]^2 / 2\bar{\sigma}^2 t}. \tag{2.91}$$

To compute the first passage time density $f(H,S_0,t)$, we have to work out the derivative of α_N, β_N and $c(\omega(\mathbf{k}))$ with respect to time t. As defined above, $\alpha_N = \prod_{i=1}^N p_i$, $\beta_N = \prod_{i=1}^N q_i$ and $c(\omega(\mathbf{k})) = \prod_{i=1}^N m^{(k_i)}$. Using the logarithm, one can find the first derivative of α_N, β_N, and $c(\omega(\mathbf{k}))$ as follows:

$$\frac{\partial \alpha_N}{\partial t} = \alpha_N \sum_{i=1}^N \frac{1}{p_i}\frac{\partial p_i}{\partial t}, \tag{2.92}$$

$$\frac{\partial \beta_N}{\partial t} = \beta_N \sum_{i=1}^N \frac{1}{q_i}\frac{\partial q_i}{\partial t}, \tag{2.93}$$

$$\frac{\partial c(\omega(\mathbf{k}))}{\partial t} = c(\omega(\mathbf{k})) \sum_{i=1}^N \frac{1}{m^{(k_i)}}\frac{\partial m^{(k_i)}}{\partial t}. \tag{2.94}$$

Substituting (2.92), (2.93) and (2.94) into (2.88) gives

$$\begin{aligned} f(H,S_0,t) &= \alpha_N \sum_{i=1}^N \frac{1}{p_i}\frac{\partial p_i}{\partial t}\Phi^{(0)}(H,S_0,t) + \alpha_N f^{(0)}(H,S_0,t) \\ &+ \beta_N \sum_{i=1}^N \frac{1}{q_i}\frac{\partial q_i}{\partial t}\Phi^{(1)}(H,S_0,t) + \beta_N f^{(1)}(H,S_0,t) \\ &+ \sum_{\omega(\mathbf{k})}\left(\sum_{i=1}^N \frac{1}{m^{(k_i)}}\frac{\partial m^{(k_i)}}{\partial t}\right) c(\omega(\mathbf{k}))\Phi^{(\{\omega(\mathbf{k})\})}(H,S_0,t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) f^{(\{\omega(\mathbf{k})\})}(H,S_0,t). \end{aligned} \tag{2.95}$$

p_i, q_i, and $m^{(k_i)}$ are given explicitly as:

$$p_i = p_0^{(i)}; \;\; q_i = p_1^{(i)} \tag{2.96}$$

and

$$m^{(k_i)} \equiv \begin{cases} p_i, & k_i{=}0; \\ q_i, & k_i{=}1. \end{cases} \tag{2.97}$$

Hence from (2.30) and (2.31)

$$\frac{\partial p_i}{\partial t} = \frac{\pi_1\lambda_1 i}{N} e^{-(\lambda_0+\lambda_1)\frac{it}{N}} - \frac{\pi_0\lambda_0 i}{N} e^{-(\lambda_0+\lambda_1)\frac{it}{N}} = \frac{i(\pi_1\lambda_1 - \pi_0\lambda_0)}{N} e^{-(\lambda_0+\lambda_1)\frac{it}{N}}, \tag{2.98}$$

$$\frac{\partial q_i}{\partial t} = \frac{\pi_0\lambda_0 i}{N}e^{-(\lambda_0+\lambda_1)\frac{it}{N}} - \frac{\pi_1\lambda_1 i}{N}e^{-(\lambda_0+\lambda_1)\frac{it}{N}} = \frac{i(\pi_0\lambda_0 - \pi_1\lambda_1)}{N}e^{-(\lambda_0+\lambda_1)\frac{it}{N}}, \quad (2.99)$$

and

$$\frac{\partial m^{(k_i)}}{\partial t} \equiv \begin{cases} \frac{\partial p_i}{\partial t}, & k_i=0; \\ \frac{\partial q_i}{\partial t}, & k_i=1. \end{cases} \quad (2.100)$$

Then (2.95) becomes

$$\begin{aligned} f(H, S_0, t) &= \alpha_N \Phi^{(0)}(H, S_0, t) \sum_{i=1}^{N} \frac{i(\pi_1\lambda_1 - \pi_0\lambda_0)}{p_i N} e^{-(\lambda_0+\lambda_1)\frac{it}{N}} + \alpha_N f^{(0)}(H, S_0, t) \\ &+ \beta_N \Phi^{(1)}(H, S_0, t) \sum_{i=1}^{N} \frac{i(\pi_0\lambda_0 - \pi_1\lambda_1)}{q_i N} e^{-(\lambda_0+\lambda_1)\frac{it}{N}} + \beta_N f^{(1)}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} \left(\sum_{i=1}^{N} \frac{1}{m^{(k_i)}} \frac{\partial m^{(k_i)}}{\partial t} \right) c(\omega(\mathbf{k})) \Phi^{(\{\omega(\mathbf{k})\})}(H, S_0, t) \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k})) f^{(\{\omega(\mathbf{k})\})}(H, S_0, t). \end{aligned} \quad (2.101)$$

2.6.2 Extrema of Geometric Brownian Motion with Markov Switching

Now we consider the minima and maxima of the geometric Brownian motion with Markov switching. We hence define the maximum and minimum of the process as

$$M_s(t) \equiv \max_{0 \le \tau \le t} S_\tau; \quad m_s(t) \equiv \min_{0 \le \tau \le t} S_\tau. \quad (2.102)$$

The probability that the maximum of the process is greater than a certain value $S \ge S_0$ is given by

$$P(M_s(t) \ge S) = P(\tau_s \le t) = \Phi(S, S_0, t) \quad (2.103)$$

where $\Phi(S, S_0, t)$is given in(2.76) while replacing H by S. The corresponding cumulative distribution function for $M_s(t)$ is

$$\Phi_M(S \mid S_0, t) = 1 - \Phi(S, S_0, t), \quad S_0 \le S \quad (2.104)$$

and the corresponding probability density function is

$$p_M(S|S_0, t) = \frac{-\partial\Phi(S, S_0, t)}{\partial S}, \quad S_0 \le S. \tag{2.105}$$

Similarly, the c.d.f. and p.d.f. for the minimum $m_s(t)$ are

$$\Phi_m(S \mid S_0, t) = \Phi(S, S_0, t), \quad S_0 \ge S, \tag{2.106}$$

$$p_m(S|S_0, t) = \frac{\partial\Phi(S, S_0, t)}{\partial S}, \quad S_0 \ge S. \tag{2.107}$$

In this case $\Phi(S, S_0, t)$ is given by (2.77) while replacing H by S.

2.6.3 Pricing Lookback Options

Based on the Section 2.6.2 we can now price lookback options whose payoff depends on the extrema of an asset path and possibly its terminal value. Here we still assume that sampling of asset prices begins at current time $t = 0$ (i.e., at contract inception) and end at maturity time $t = T$. The extension that accounts for sampling prior to current time is trivial.

Assume a money market account of worth $B_t = e^{rt}$, and a floating strike put with payoff function $P_T = (M_s(T) - S_T)_+$. Under the risk-neutral measure Q with B_t as numeraire, the corresponding lookback option price , with the time-to-maturity T, is given by the following

$$P_0(S_0, T) = e^{-rT} E_0^Q[M_s(T)] - e^{-rT} E_0^Q[S_T] = e^{-rT} E_0^Q[M_s(T)] - S_0. \tag{2.108}$$

In order to compute the lookback option price, we must hence compute $E_0^Q[M_s(T)]$. This is the risk-neutral expectation of the maximum conditional on current asset price S_0, and is given as follows:

$$\begin{aligned} E_0^Q[M_s(T)] &= \int_{S_0}^{+\infty} p_M(S|S_0, T) S dS = \int_{S_0}^{+\infty} \frac{-\partial\Phi(S, S_0, T)}{\partial S} S dS \\ &= -S\Phi(S, S_0, T) \mid_{S_0}^{\infty} + \int_{S_0}^{\infty} \Phi(S, S_0, T) dS \\ &= S_0 + \int_{S_0}^{\infty} \Phi(S, S_0, T) dS. \end{aligned} \tag{2.109}$$

Here we used $\lim_{S\to\infty} S\Phi(S,S_0,T) = 0$ and $\Phi(S_0,S_0,T) = 1$ and integration by parts. Substitute (2.109) into (2.108) and using (2.77), we have

$$\begin{aligned} P_0(S_0,T) &= e^{-rT}\int_{S_0}^{\infty}\Phi(S,S_0,T)dS + S_0[e^{-rT}-1] \\ &= e^{-rT}\bigg[\alpha_N\int_{S_0}^{\infty}\Phi^{(0)}(S,S_0,T)dS + \beta_N\int_{S_0}^{\infty}\Phi^{(1)}(S,S_0,T)dS \\ &+ \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k}))\int_{S_0}^{\infty}\Phi^{(\{\omega(\mathbf{k})\})}(S,S_0,T)dS\bigg] + S_0[e^{-rT}-1]. \quad (2.110) \end{aligned}$$

From the classical result of lookback option pricing under geometric Brownian motion $dS_t = rS_t dt + \sigma S_t dB_t$, we know that the floating strike lookback value is:

$$p_0(S_0,T) = S_0 e^{-rT}\left(1-\frac{\sigma^2}{2r}\right)N(\tilde{d}_-) - S_0\left(N(-\tilde{d}_+) - \frac{\sigma^2}{2r}N(\tilde{d}_+)\right), \quad (2.111)$$

where

$$\tilde{d}_\pm = (\frac{1}{2} \pm \frac{r}{\sigma^2})\sigma\sqrt{T}. \quad (2.112)$$

Let us denote the lookback prices for the respective GBM regime by

$$p_0^{(0)} = e^{-rT}\int_{S_0}^{\infty}\Phi^{(0)}(S,S_0,T)dS + S_0[e^{-rT}-1], \quad (2.113)$$

$$p_0^{(1)} = e^{-rT}\int_{S_0}^{\infty}\Phi^{(1)}(S,S_0,T)dS + S_0[e^{-rT}-1], \quad (2.114)$$

and

$$p_0^{(\{\omega(\mathbf{k})\})} = e^{-rT}\int_{S_0}^{\infty}\Phi^{(\{\omega(\mathbf{k})\})}(S,S_0,T)dS + S_0[e^{-rT}-1]. \quad (2.115)$$

$p_0^{(0)}$, $p_0^{(1)}$, $p_0^{(\{\omega(\mathbf{k})\})}$ are obtained by applying the resulting formula given in (2.111) with the appropriate volatility $\sigma = \sigma_0, \sigma_1$, or $\bar{\sigma}$. Thus from (2.78) it follows that, the pricing formula for the lookback option in our example (under Markov switching GBM) is

$$P_0(S_0,T) = \alpha_N p_0^{(0)} + \beta_N p_0^{(1)} + \sum_{\omega(\mathbf{k})} c(\omega(\mathbf{k}))p_0^{(\{\omega(\mathbf{k})\})} \quad (2.116)$$

where

$$p_0^{(0)}(S_0,T) = S_0 e^{-rT}\left(1-\frac{\sigma_0^2}{2r}\right)N(\tilde{d}_-) - S_0\left(N(-\tilde{d}_+) - \frac{\sigma_0^2}{2r}N(\tilde{d}_+)\right), \quad (2.117)$$

$$p_0^{(1)}(S_0,T) = S_0 e^{-rT}\left(1-\frac{\sigma_1^2}{2r}\right)N(\tilde{d}_-) - S_0\left(N(-\tilde{d}_+) - \frac{\sigma_1^2}{2r}N(\tilde{d}_+)\right), \quad (2.118)$$

and

$$p_0^{(\{\omega(\mathbf{k})\})}(S_0,T) = S_0 e^{-rT}\left(1-\frac{\bar{\sigma}^2}{2r}\right)N(\tilde{d}_-) - S_0\left(N(-\tilde{d}_+) - \frac{\bar{\sigma}^2}{2r}N(\tilde{d}_+)\right). \, (2.119)$$

$\tilde{d}_\pm$ is given in (2.112) with the replacement of proper choice of σ. The pricing formula is also a linear combination of the resulting option price formula under GBM models. Analytical formulas for other types of put or call-like lookbacks also follow in similar fashion.

Chapter 3

Calibration and Sensitivity Analysis

In this Chapter we obtain numerical results for the Markov switching model with lognormal percentage of jump size as an example. The numerical study of other models is the subject of future work. The method used in other models is similar with the case that we are discussing.

3.1 Model Calibration

3.1.1 Newton Method for Non-linear Optimization Problem

Given that all the parameters in the model have been specified, the computation of option prices under our new model gives us the opportunity to produce the corresponding implied volatility surface, if we assume that market prices are computed according to our model. By setting the Black-Scholes option price to be equal to the market price at the point (K, T), for given strike K and maturity T, we can compute the implied local volatility at this data point using a numerical method.

We define

$$k(\sigma) = m(\sigma) - C$$

where C is the market price (or model price) and $m(\sigma)$ is the price under B-S model with given volatility σ. Therefore, the first order Taylor expansion around $\sigma = \sigma^*$ is

$$k(\sigma) \approx k(\sigma^*) + k'(\sigma^*)(\sigma - \sigma^*) = 0.$$

Hence

$$\begin{aligned} \sigma &= \sigma^* - \frac{k(\sigma^*)}{k'(\sigma^*)} \\ &= \sigma^* - \frac{m(\sigma^*) - C}{m'(\sigma^*)}. \end{aligned} \tag{3.1}$$

Replacing σ and σ^* by $\sigma^{(i)}$ and $\sigma^{(i-1)}$, we have the following iteration

$$\sigma^{(i)} = \sigma^{(i-1)} - \frac{m(\sigma^{(i-1)}) - C}{m'(\sigma^{(i-1)})}.$$

In order to use the Newton method, the first derivative of the given function has to be evaluated. For the option price, the first derivative with respect to σ is the so-called Vega within the B-S mode, it is defined as:

$$V(\sigma) = e^{-rT}\Big(- F(\sigma d_1 - \sqrt{T})\phi(d_1) + K\sigma d_1 \phi(d_2)\Big). \tag{3.2}$$

In particular, where F is the forward price and is equal to $S_0 e^{rT}$ and $\phi(\cdot)$ is normal density. Using Newton's method, the iteration formula is given by

$$\sigma_i = \sigma_{i-1} - \frac{m(\sigma_{i-1}) - C}{V(\sigma_{i-1})}. \tag{3.3}$$

In general, we will have a problem to find the solution for σ if we are searching in a wider interval. It is almost not possible to find a global minimum or maximum value by using Newton's method if there exists a local minimum or local maximum.

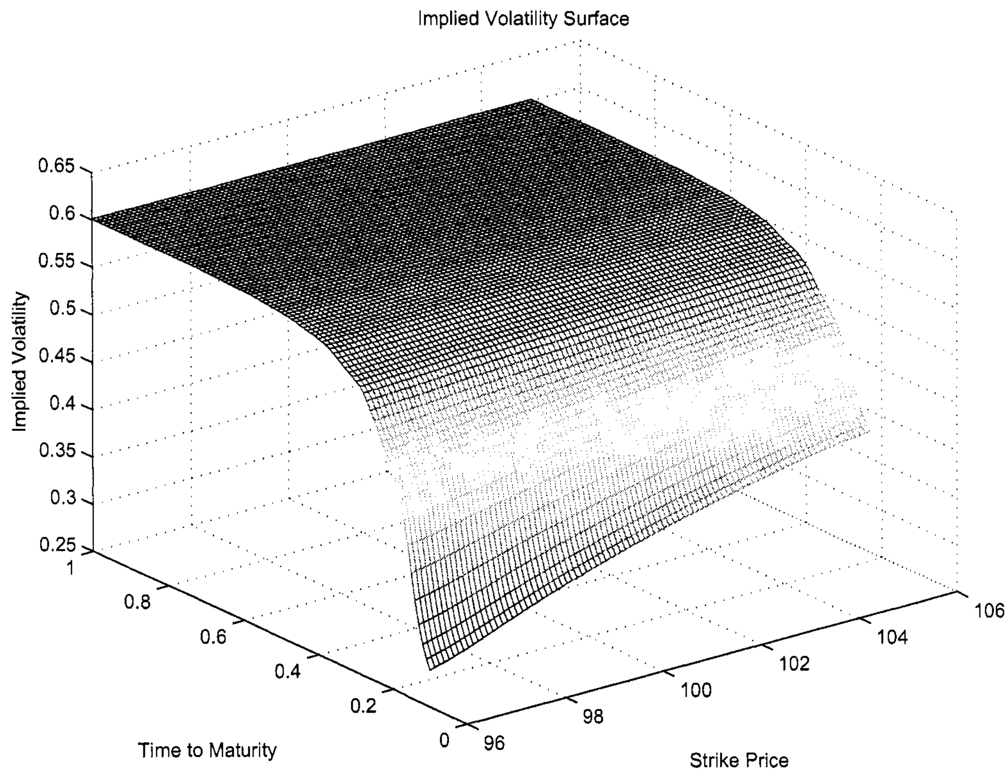

Figure 3.1: Implied Volatility Surface with Higher Current Market Volatility.

So we should pay particular attention in this application. The way we may be successful is to set a smaller searching interval for strike price and time to maturity when we produce an implied volatility surface. Note: if the market price using this model is underpriced, we may not be able to find the implied volatility.

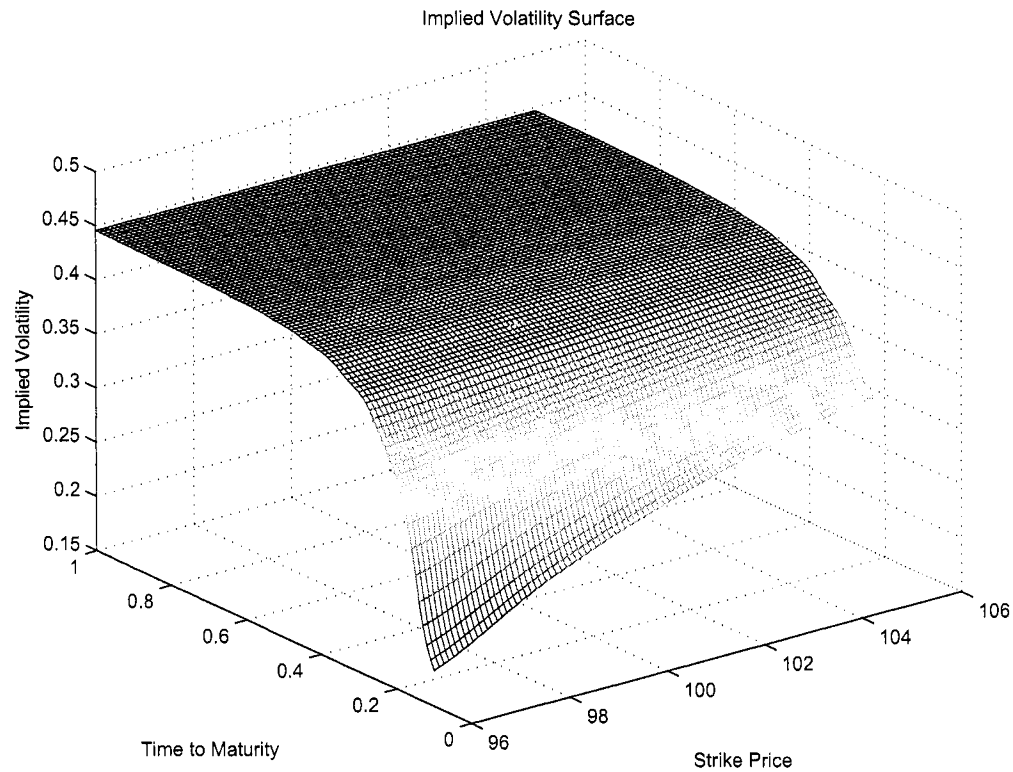

Figure 3.2: Implied Volatility Surface with Lower Current Market Volatility.

3.1.2 Kernel Smoothing Technique

Current research [10] presents a numerical method to develop the dynamic representation of implied volatility surfaces, which relies on the correlation matrix for a given implied volatility surface. In order to apply the Karhunen-Loeve decomposition of the daily variations of implied volatilities obtained from market data, we need to construct such a correlation matrix. Therefore, it is important to have sufficient data points on the surface to more precisely compute the correlation matrix. We rely on kernel smoothing techniques to compute the implied volatility given the combination of the strike price and maturity time. Essentially, this technique uses a smoothing kernel function to connect all the data points within the given bandwidth. In this thesis, a two dimensional Gaussian kernel function has been

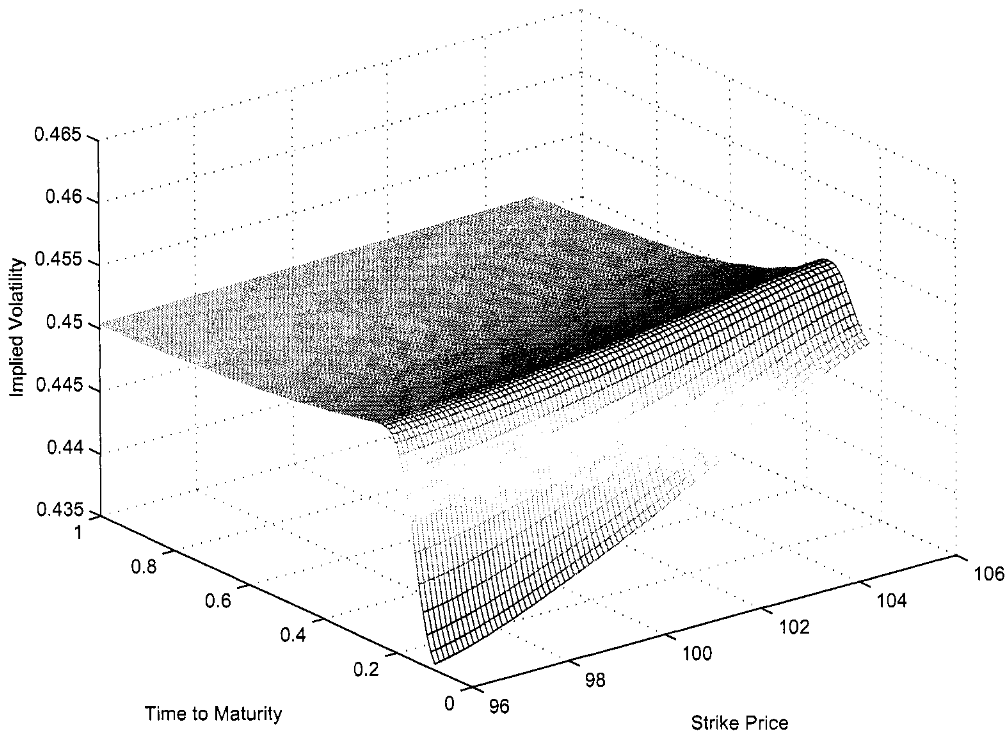

Figure 3.3: Implied Volatility Surface with Higher Switching Intensity in Current State.

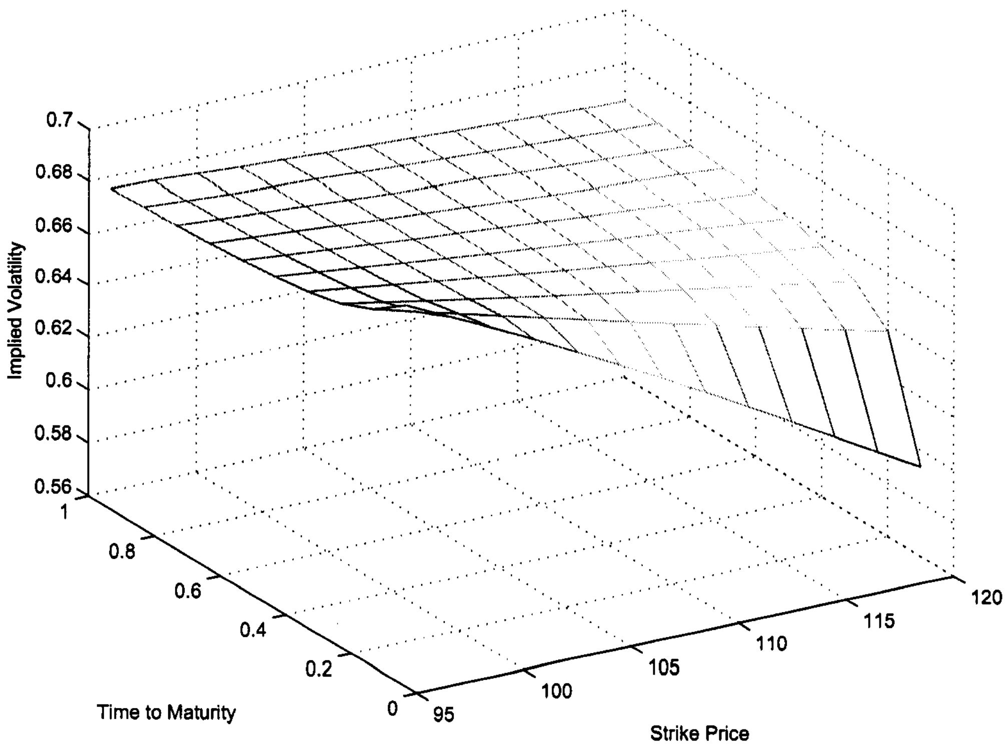

Figure 3.4: Implied Volatility Surface with μ_J=-0.1.

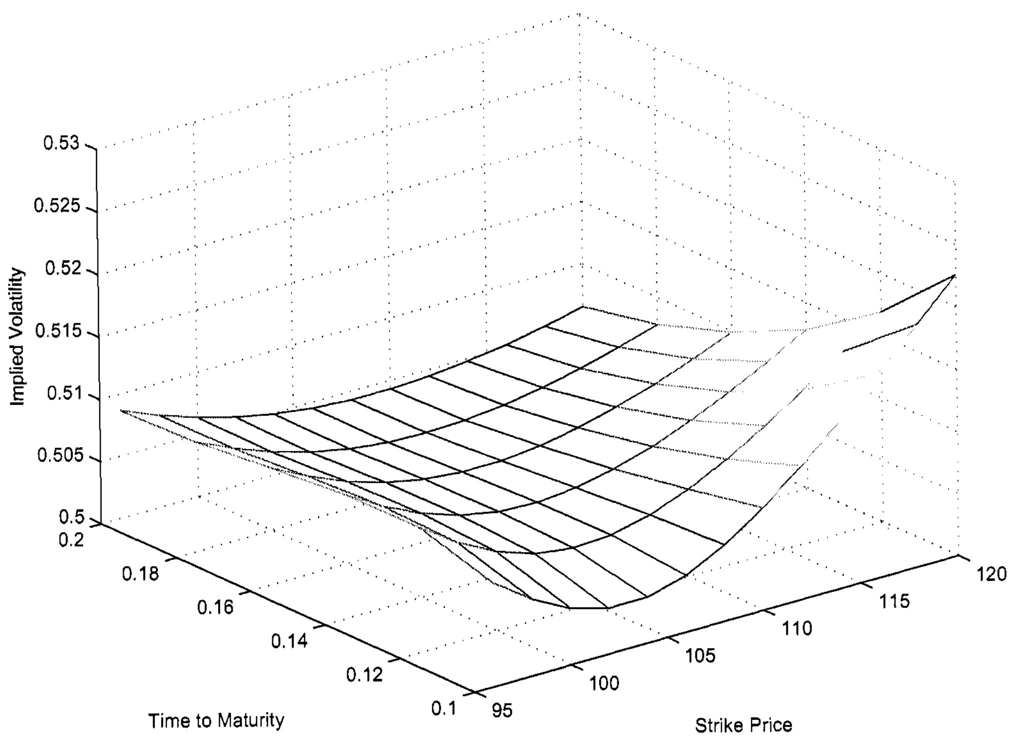

Figure 3.5: Smile Implied Volatility Surface with μ_J=-0.01 and Shorter Maturity Time.

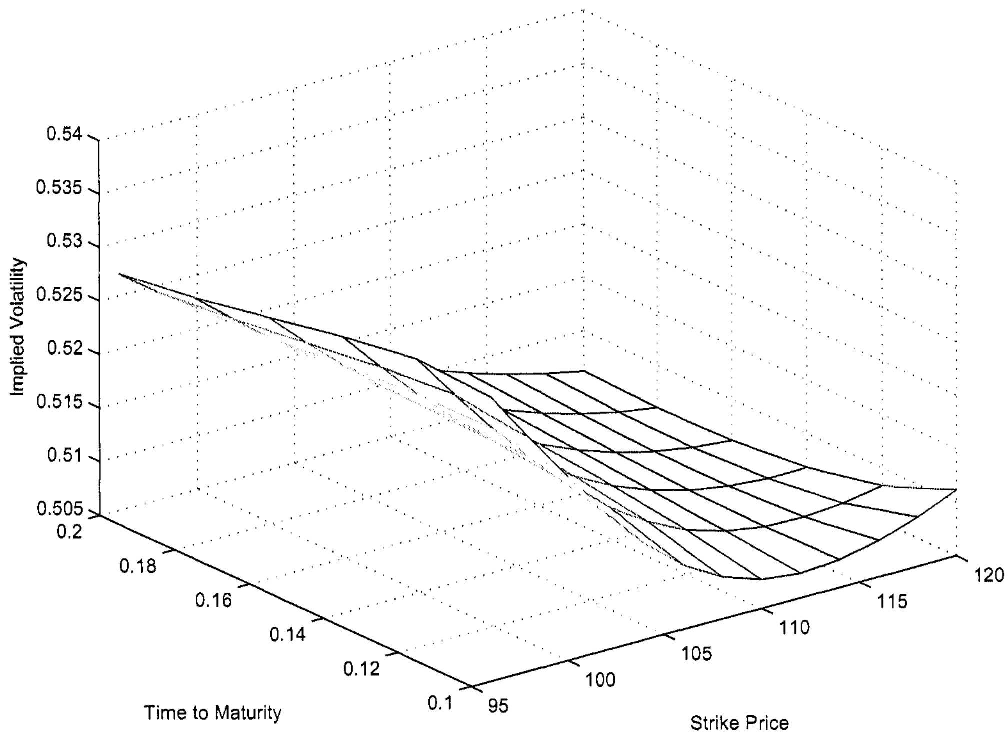

Figure 3.6: Smile Implied Volatility Surface with μ_J=-0.03 and Shorter Maturity Time.

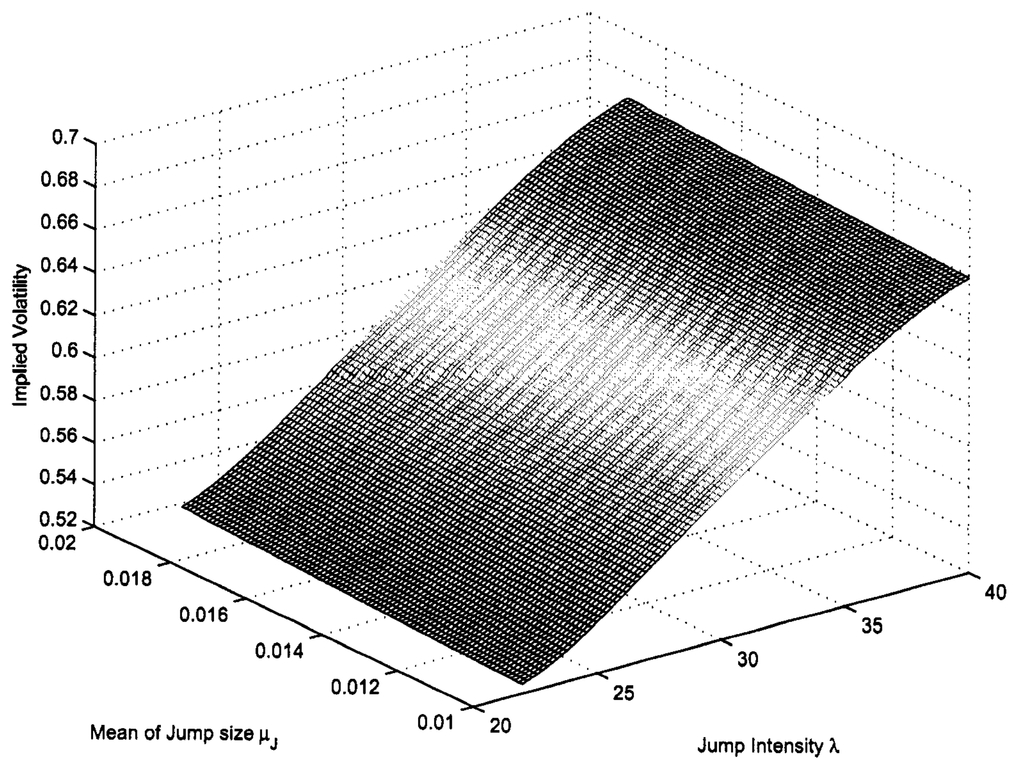

Figure 3.7: Implied Volatility Surface with Changing Mean of Jump Size and Intensity of Jump.

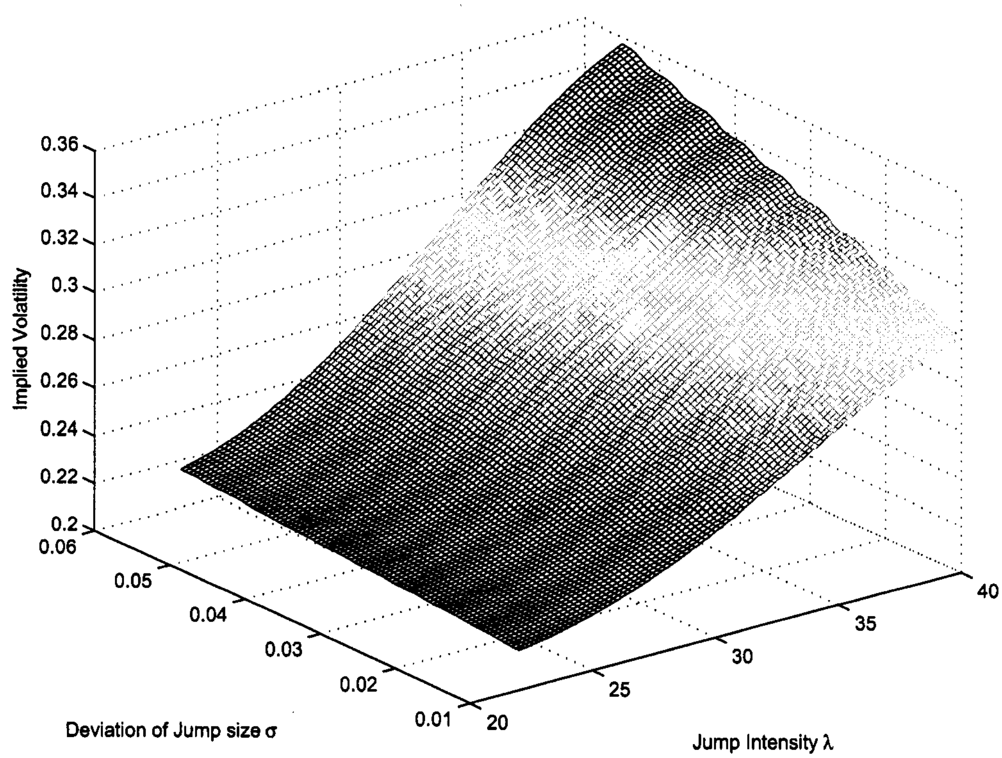

Figure 3.8: Implied Volatility Surface with Changing Deviation of Jump Size and Intensity of Jump.

used:

$$g(x,y) = \frac{1}{2\pi\sqrt{h_1 h_2}} \exp\left\{\frac{-(x^2+y^2)}{4h_1h_2}\right\}. \tag{3.4}$$

Using this Gaussian kernel function, a non-parametric Nadaraya-Watson estimator is obtained as follows:

$$\hat{I}_t(m,\tau) = \frac{\sum_{i=1}^{n} I_t(m_i,\tau_i) g(m-m_i, \tau-\tau_i)}{\sum_{i=1}^{n} g(m-m_i,\tau-\tau_i)}. \tag{3.5}$$

By applying the non-parametric estimator, we produce a smooth surface of the implied volatility. The main problem with this non-parametric estimator is how to choose the optimal bandwidth to fit the data. Generally speaking, we prefer the bandwidth to be as small as possible, but the small bandwidth will produce a bumpy surface, which is not in our interest. In practice, we could start choosing a smaller bandwidth. At this stage, we may or may not have bumpy surface problems, but we would like to choose a bandwidth that has a bumpy surface, then increase the bandwidth by a small step until we get rid of the bumps.

3.1.3 Comparison of Option Prices

For this price comparison study, we set the following value of parameters for our J-M (Jump Markov switching) model: risk free interest rate, $r = 0.03$; initial stock price, $S_0 = 100$; jump intensity, $\lambda = 15$; intensity of remaining in state 0, $\lambda_0 = 10$; intensity of remaining in state 1, $\lambda_1 = 50$; mean of jump size process, $\mu_J = 0.01$; standard deviation of jump size, $\sigma_J = 0.1$; volatility of regime 0, $\sigma_0 = 0.1$ and volatility of regime 1, $\sigma_1 = 0.5$; The σ shown in Table 3.1 is the volatility used to compute prices in the B-S model. From the comparisons in Table 3.1, the J-M model gives option values between the option price using B-S model with $\sigma = 0.1$ and $\sigma = 0.5$. When σ=0.1, K=96, T=0.1, the B-S model give an option price of 4.4 and when σ=0.5, K=103, T=1, the B-S model give an option price of 19.76. This tells us that the Markov switching Jump Diffusion model will produce a range

Table 3.1: Comparison of option prices between the Black-Scholes model and the Markov switching Jump Diffusion(J-M) model.

	B-S model	J-M model
K=96, T=0.1, σ=0.4	7.37	4.77
K=97, T=0.2, σ=0.4	8.90	8.27
K=98, T=0.3, σ=0.4	10.11	10.41
K=99, T=0.4, σ=0.4	11.08	11.88
K=100, T=0.5, σ=0.45	13.31	13.02
K=101, T=0.6, σ=0.45	14.18	13.97
K=102, T=0.7, σ=0.45	14.98	14.80
K=103, T=0.8, σ=0.45	15.72	15.55
K=104, T=0.9, σ=0.45	16.40	16.25
K=105, T=1, σ=0.45	17.04	16.90

of market prices in between for certain jump parameters(as shown in Figure 3.9). Note: The European call option price for σ_0 and σ_1 are not hard bounds for the Markov switching jump model when the jump parameters are varied. The J-M model is hence more flexible, allowing a choice of proper parameters for the model in the model calibration.

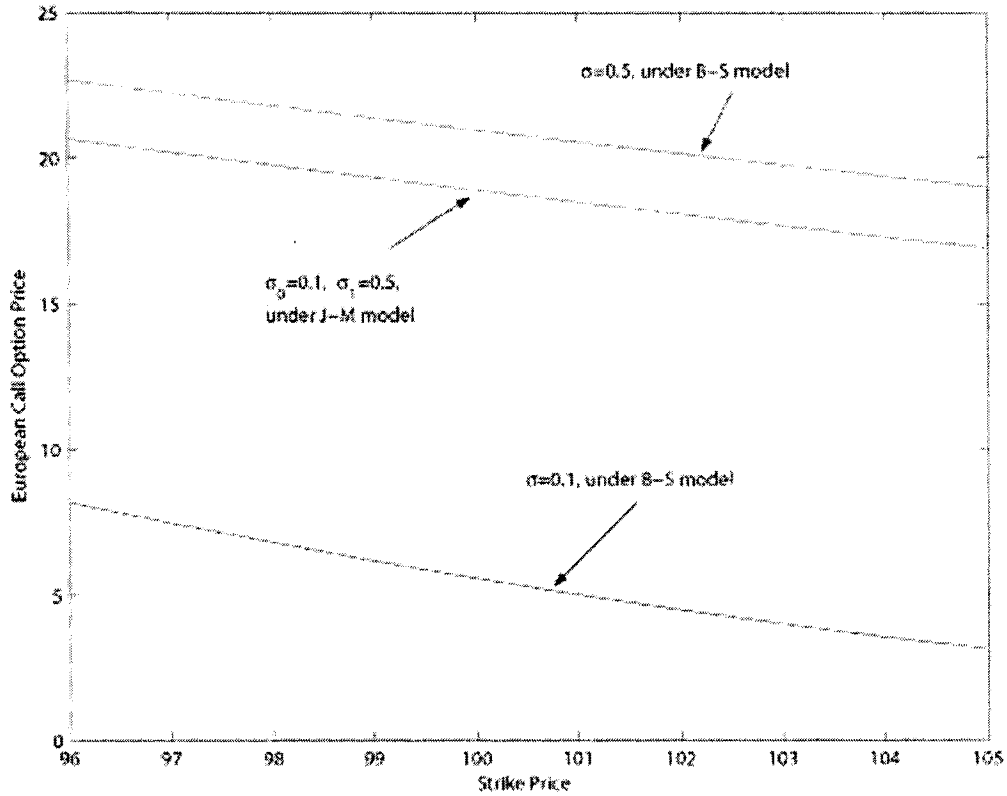

Figure 3.9: European Call Option Price Comparison under B-S and J-M Models (T=1).

3.1.4 Implied Volatility Surface

As an illustration of option pricing using the Markov switching Jump Diffusion model, we consider the case with current market status known in advance. Throughout the entire calibration study, we assume that $r = 0.03$, $S_0 = 100$. In Figure 3.1 we set jump intensity, $\lambda = 15$; intensity of remaining in state 0, $\lambda_0 = 10$; intensity of remaining in state 1, $\lambda_1 = 50$; mean of jump size process, $\mu_J = 0.01$; deviation of jump size, $\sigma_J = 0.1$; volatility of regime 0, $\sigma_0 = 0.5$ and volatility of regime 1, $\sigma_1 = 0.1$; When the state is in the lower switching intensity and the current state of volatility is higher($\sigma_0 = 0.5$ in this case), we obtain an implied volatility surface with volatility varying from 0.26 to 0.6. As we reset the value of σ_0 to be 0.1 and σ_1 to be 0.5, the volatility varies from 0.15 to 0.45. We can see the details in Figure 3.2. If we switch the value of λ_0 and λ_1 in first scenario and keep other parameters unchanged, we can produce another kind of result given in Figure 3.3. This shows less changes in the value of the volatility: from 0.43 to 0.45. We can see from here that if the current state is in higher volatility and also with a high intensity of remaining in current state, then the implied volatility for the model remains in a similar range. We can produce different shapes of the implied volatility surface including the smile surface by choosing the proper parameters. (This is shown in Figure 3.1 - 3.6). Also, as T gets larger, the implied volatility surface becomes flatter. This result can be understood by using (1.18), as this limiting transition matrix gives the asymptotic result for long run effects of Markov switching.

3.1.5 Calibration to Real Data

As we can see from Figures 1.5 and 1.6, the shape of the implied volatility surface is similar to the ones that are shown in Figure 3.1 and Figure 3.2. We can calibrate the model parameters to an observed implied volatility surface using market prices, such as S&P500 option prices. We can use a least squares best fit, by optimizing the model parameters that minimize:

$$\min_{\theta} \sum_{j=1}^{N} (C_j - C_j(\theta))^2 \tag{3.6}$$

where N is total number of observed market prices over a range of (K, T) values, C_j stands for an observed market price, theta is the parameter space and $C_j(\theta)$ is the corresponding model price. Such a calibration is generally computationally intensive since the model parameter space is considerably larger than in the B-S model. To simplify the calculations, we may consider fixing a subset of the more insensitive parameters to appropriate constants, such as the percentage of jump distribution parameters (See next section). However, this important optimization problem still remains as a future work.

3.2 Sensitivity of Implied Volatility

The plots shown in Figure 3.7 and Figure 3.8 give the sensitivity of implied volatility as the parameters of jump size distribution and jump intensity are varied. From the plots, we can easily see that there is a significant change when the jump intensity is increased. On the other hand, the change in the implied volatility is negligible when we increase the mean of jump size or standard deviation of jump size. As we can see from the outputs, in order to capture some significant variation of market price, the proper choices of jump size distribution and its associated parameters should be considered. However, the intensity of the Poisson process is a major factor affecting the implied volatility.

Chapter 4

Actuarial Application

In Actuarial Science literature [12], a classical risk process is modeled by the form of $U_t = U_0 + Ct - S(t)$ for the continuous time model, where U_t is called the surplus process, U_0 is initial surplus, C is the premium rate and $S(t)$ is the total claims amount. $S(t)$ is the aggregate loss described by a compound Poisson process as usual. This classical risk model is usually studied using renewal theory, Markov chain and random walks. Those theories and techniques are very useful for this classical risk model assuming a certain type of severity distribution. However, there are some imperfections inherent in this model. First of all, the model does not consider the growth of premium which is very significant when the investment return or cumulative interest rate is high. On the other hand, the premium rate should be time dependent. Auto insurance policies normally are issued for a one year period and renewal for the future year in Canada. If the insured has a large amount of claim during the fiscal year, the renewal premium will normally rise significantly. So, when the insured renews the insurance policy, he or she has to pay a higher rate of premium since the risk class has been changed due to the insurance claims. This means that the loss for the current year will be made up by increasing future premium rates for those individuals under the assumption that the real market is incomplete.

But even in a complete market, we also have the same problem regarding the insurance premium. Current research [8] is showing that the insurance market has been subject to pricing cycles and it has been categorized into two type of markets: soft market and hard market. A soft market means the sum of premium goals for all insurance companies operating in the given market is greater than the total amount of insurance desired by all potential insureds in that market. A hard market means the sum of premium goals for insurance companies operating in a given market is lower than the amount of insurance desired by all potential insureds in that market. In a soft market, insurance companies might select to sign insurance contracts during the underwriting period even though they may suffer a significant loss later. The reason why they choose this business strategy is that one believes that it is necessary to keep the current insureds or pursue a new market share. In a hard market, insurance prices are relatively high. During the period of a hard market, insurance companies are more likely to pursue premium gains instead of market share because of the easier achievement of this goal. Therefore a mathematical model that describes this risk should reflect this important phenomenon. It is hence necessary to consider at least two different regimes of volatility of surplus (Note: the volatility regimes do not translate exactly into the hard and soft markets. We consider the lower risk for a hard market and the larger risk in a soft market, assuming the same expectation of rate of return for both).

Finally, the claims payout is delayed as usual. The classical risk model does not consider the inflation of the claim amount. Obviously, the consideration of the inflationary loss severity is very important to estimate the total losses for a company in order to analyze and forecast the future pattern of surplus. In this session, we propose a Markov switching risk model that can be used for future decision making and surplus pattern forecasting. It is more reasonable to use the

Markov switching model from a practical point of view.

4.1 Modelling the Risk Process

4.1.1 Model Specification

Consider the following model

$$\frac{dU_t}{U_{t-}} = rdt + \sigma_{\epsilon_t} dB_t - d\Bigg(\sum_{i=1}^{N(t)}(V_t + 1)\Bigg), \tag{4.1}$$

where U_t is recognized as the surplus of the insurance company at time t, r is the instantaneous growth rate of the company's assets, σ_{ϵ_t} is the volatility of the growth for specific state ϵ_t, V_t is the ratio of the surplus before jump and the surplus after jump and $N(t)$ is the claim frequency random variable. As is already known, the surplus is usually a discrete random variable since the observations cannot be taken frequently. But it is necessary to consider surplus as continuous except at the jump point from a mathematical point of view. Without the term $d\Big(\sum_{i=1}^{N(t)}(V_t+1)\Big)$, we know the solution of U_t is

$$U_t = U_0 e^{(r - \frac{1}{2}\sigma_{\epsilon_t}^2)t + \sigma_{\epsilon_t} B_t} \tag{4.2}$$

for the associated SDE, where U_0 is the initial reserve. Since the operation of a company is much more complicated in the real world, we need some simplification on the growth rate of assets, in order to estimate the overall effect on the company's growth. The following factors should be taken into account: premiums, investment return of assets and business operation expenses. In our model, we do not focus on how those factors affect the surplus process; instead, we only consider the overall effect from those factors.

To make a connection between the Markov switching risk model and the classical risk model, we need to know the distribution of V_t as we only know the distribution of loss severity rather than the percentage rate of loss severity. Here, we define

$$Y_t = \int_0^t (V_s + 1) dN_s = \sum_{i=1}^{N(t)} (V_t + 1). \tag{4.3}$$

Note that $X_t = U_{t-}(V_t+1)$, where X_t is a claim amount at time t, we can determine what is V_t in our model. From the definition of X_t, we know V_t satisfies

$$V_t = \frac{X_t}{U_{t-}} - 1, \; U_{t-} > 0. \tag{4.4}$$

If we observe a claim amount X_t and surplus immediately before time t, then V_t is obtained:

$$V_t = \frac{X_t}{U_{t-}} - 1 = \frac{X_t - U_{t-}}{U_{t-}} = -\frac{U_t}{U_{t-}}, \tag{4.5}$$

where we have defined $U_{t-} - U_t = X_t$. Here we simplify our model by only considering the case that any growth effect lags any claim effect on a company's surplus. (This is true for insurance since the premium is normally paid at the time of validation of the insurance policy if we only consider the case of premium effects). Moreover, if lognormality is assumed, we have -$V_t \sim$ lognormal. Denote $M_t = -V_t$, then M_t is lognormal and $\ln M_t$ follows a normal distribution. Yet we still need to know under what distribution of loss severity will, M_t follow a lognormal distribution. In next section, we will discuss when the lognormality should be assumed. The explanation of the Markov switching risk model and its associated parameters help us determine the connection between these two models.

4.1.2 Distribution of V_t under Lognormal Loss Severity

From the discussions above, we should know what distribution V_t could be if we know the distribution of U_t and X_t. As is known U_t has a jump at the start of a certain period and U_t is a continuous random variable in the interval of claims arrival. Suppose that ξ_1 and ξ_2 are random variables. If ξ_1 has density f_{ξ_1} and ξ_2 has density f_{ξ_2} and are assumed independent, then the density of the ratio $\frac{\xi_1}{\xi_2}$ is given by

$$f_{\frac{\xi_1}{\xi_2}}(x) = \int_{-\infty}^{\infty} f_{\xi_1}(x_2 x) f_{\xi_2}(x_2) \mid x_2 \mid dx_2, \tag{4.6}$$

We know U_t before a jump has lognormal distribution with parameters μ_1 and σ_1. Furthermore, if we assume the claims severity X_t also has lognormal distribution with parameter μ_2 and σ_2. Then

$$f_{\frac{\xi_1}{\xi_2}}(z) = \int_0^{\infty} \frac{1}{yz\sigma_1\sqrt{2\pi}} \exp\left\{ -\frac{(\ln(yz)-\mu_1)^2}{2\sigma_1^2} \right\} \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left\{ -\frac{(\ln y-\mu_2)^2}{2\sigma_2^2} \right\} dy.$$

Changing integration variable and using the identity of (2.45), we easily obtain that the density function for $\frac{\xi_1}{\xi_2}$ is lognormal. Therefore, if we have lognormal loss severity X_t, then the lognormality of V_t is obtained. In other words, lognormality of X_t is equivalent with lognormality of V_t and different loss severity distributions will create different distributions for V_t.

4.2 Ruin Probability for Risk Model

4.2.1 Without Markov Switching

When there is no switching of volatility σ, we can apply the Itô formula for jump diffusions, and it is not difficult to verify that the solution to (4.1), where $\sigma_{\epsilon_t} = \sigma =$ constant, is

$$U_t = U_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t} \Pi_{i=1}^{N(t)} M_t. \tag{4.7}$$

Since $M_t = -V_t$ and $V_t + 1$ is the ratio of X_t and U_{t-} , if $X_t > U_{t-}$ at time t, then ruin occurs immediately. Otherwise U_t continues to grow. If we are interested in the ruin probability, which is given by $P(U_t < 0)$ for ultimate ruin and $P(U_t < 0, t < \infty)$ for finite time ruin probability, simulation based methods are a good choice for calculating those probabilities. Again, it is easy to see that $P(U_t < 0 \mid$

$t < \infty) = E(I(U_t < 0) \mid t < \infty)$ and we can calculate the expectation by Monte Carlo simulation. We need to simulate the lognormal random variable M_t along with the claim times governed by $N(t)$ and B_t from Brownian motion. Another useful method to be used to evaluate ruin probabilities is the normal approximation if we consider the probability of the surplus below some given certain value U, where $U > 0$. Note that, conditional on $N(t) = j, j \geq 0$, we obtain a lognormal distribution for U_t. Therefore

$$\begin{aligned} P(U_t < U \mid N(t) = j) &= P(U_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t} \Pi_{i=1}^{N(t)} M_t < U \mid N(t) = j) \\ &= P(\ln U_0 + (r - \frac{1}{2}\sigma^2)t + \sigma B_t + \sum_{i=1}^{j} \ln M_t < \ln U) \\ &= P(\eta_j < \ln U), \end{aligned} \tag{4.8}$$

where $\eta_j = \ln U_0 + (r - \frac{1}{2}\sigma^2)t + \sigma B_t + \sum_{i=1}^{j} \ln M_t$ is a normal random variable. If we denote its mean and variance by m_j and v_j, then $\eta_j \sim N(m_j, v_j)$. Thus

$$\begin{aligned} P(U_t < U) &= E_j(P(U_t < U \mid N(t) = j)) = E_j(P(\ln U_t < \ln U) \mid N(t) = j) \\ &= E_j\left(\Phi(\frac{\ln U - m_j}{\sqrt{v_j}})\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \Phi(\frac{\ln U - m_j}{\sqrt{v_j}}). \end{aligned} \tag{4.9}$$

This gives us an approximation of $P(U_t < U)$.

4.2.2 With Markov Switching

Typically, we consider the ruin probability given initial surplus $U_0 > U$ so that it can be represented by computing the probability $P(U_T < U)$ given the condition that $U_0 > U$. This probability is the same as the probability that the asset price process has attained a lower barrier H and has been absorbed, in this case $H = U$. Therefore the ruin probability under this stochastic risk model with Markov switching is given in (2.77) by adding the lognormal jump component into the model.

As an example, we simulate the risk process from the classical risk model with diffusion:

$$U_t = U_0 + Ct + \sigma B_t - \sum_{i=1}^{N(t)} X_i. \tag{4.10}$$

Note: This will reduce to the classical result as we set $\sigma = 0$. We also simulate the process with Markov switching and compute the finite time ruin probability in order to investigate how the switching affects the ruin probability. For the switching process we use:

$$U_t = U_0 + Ct + \sigma_{\epsilon_t} B_t - \sum_{i=1}^{N(t)} X_i \tag{4.11}$$

In this example, we assume that the loss distribution X_i follows a mixed exponential with CDF:

$$F(x) = 1 - w_0 e^{-x} - (1 - w_0)e^{-2x} \tag{4.12}$$

where $0 \leq w_0 \leq 1$. In this example, the value of w_0 is $\frac{2}{3}$. We set $\sigma = 0.5, \sigma_0 = 0.1, \sigma_1 = 0.9, \lambda = 10, \lambda_0 = 50, \lambda_1 = 50$ and time horizon $T = 5$ years. We also set the security loading $\theta = 0.2$. The comparison of ruin probabilities with respect to initial reserve U_0 is shown in Figure 4.1. It is clear from the simulations that the ruin probability is higher for both the Markov switching diffusion model and the classical risk model with diffusion. This is true since the ruin occurs not only at the time of claim arrival, but also at time of investment loss or operation loss, which is captured by the diffusion. As we increase the initial reserve, the difference of finite time ruin probability between these models is not significant. Therefore, these models will reduce to the classical risk model in the sense of ruin probability.

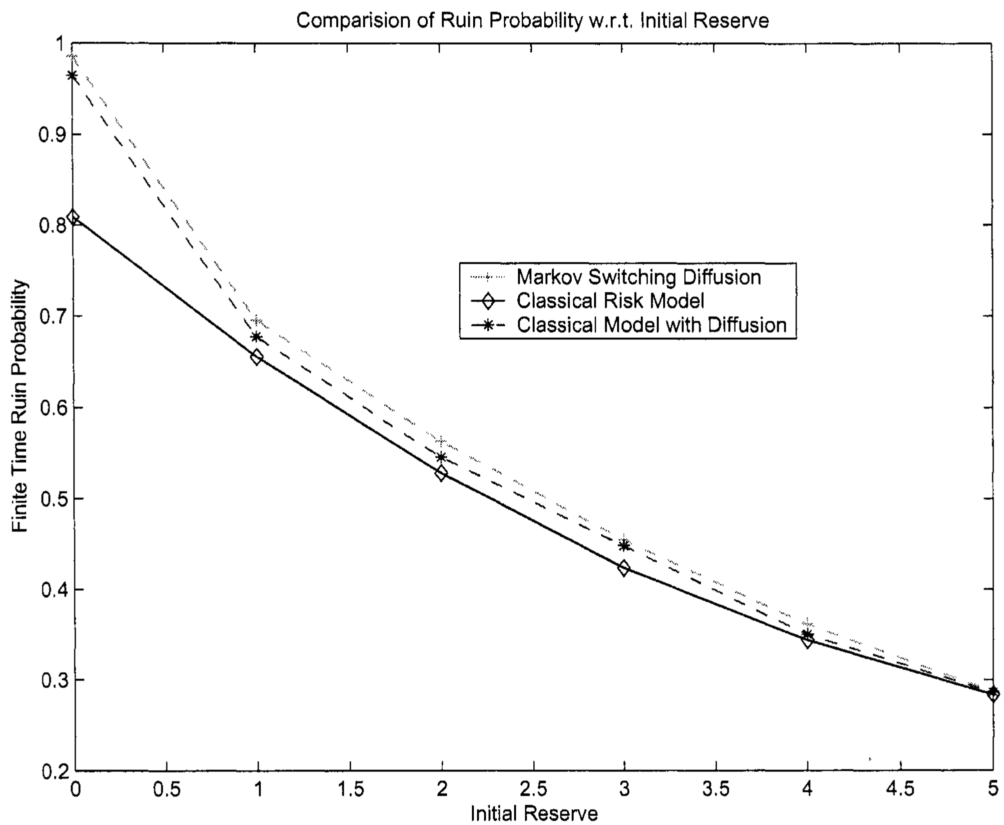

Figure 4.1: Comparison of Ruin Probability w.r.t. Initial Reserve.

Chapter 5

Conclusion and Future Work

In this thesis, we have considered extensions of Black-Scholes (B-S) model by adding Jump processes and hidden Markov chains to geometric Brownian motion. In order to come up with some closed-form solutions for option pricing problems, we only considered special choices of the Jump processes to make things tractable. The choices of percentage of jump size distribution include: exponential percentage jump size, lognormal percentage jump size and constant percentage jump size. We derived explicit option pricing formulas for these cases as weighted averages of B-S formulas. In this thesis, we also considered a path integral approach to find the transition density analytically for Markov switching geometric Brownian motion. Using this transition density function, we studied the first-passage time problem for an absorbing barrier and derived the extrema for this process. We also studied the ruin probability problem for a company by assuming that the surplus process follows a Markov switching jump diffusion model. The following are some conclusions drawn from this thesis.

- Jump diffusion with Markov switching can produce different kinds of implied volatility surfaces by choosing different model parameters. The smile implied volatility surface is attainable using the Markov switching Jump Diffusion model.

- Implied volatility is very sensitive with respect to jump intensity, but it is rather insensitive to the parameters of percentage jump size distribution.

- Jump diffusion with Markov switching model is a useful model for option pricing and other similar problems since it reflects the different regimes of volatility that are inherent in an economy.

- The form of our analytical pricing formulas suggest that option prices are some kind of weighted averages of the B-S formulas corresponding to different cumulative volatility with weights determined by the probability density for the process to remain in the current state.

- Path integral approaches are useful for studying the first-passage times and its associated problems for the geometric Brownian motion with Markov switching. The analytical solution for the transition probability density is attainable for this model with two-state Markov switching.

- Jump diffusion with Markov switching model is also useful in Actuarial Science.

In this thesis, we only considered the two-state Markov switching model. In fact, it can be extended to the multi-state case, this will be considered and investigated in my future studies. The Markov switching geometric Brownian motion model also seems promising for modelling credit risk migration and default rates. The study of the pricing problem under this model is attractive from a practical point of view. Also there are many types of option pricing formula that haven't been given under this model. There exists potential research opportunities here and I will continue to consider these problems during my PhD study as part of my research directions.

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